Math Refresher: Basic Derivative Rules

ECO 3302 – Intermediate Macroeconomics

Luis Pérez

Southern Methodist University

February 7, 2025

Abstract

This note outlines fundamental derivative rules for both univariate and multivariate functions. Please make sure to know these rules well—and understand how to apply them—as they will be important for our class. You should already be familiar with these concepts from previous coursework, including one or more of the following prerequisite courses: ECO 1311 (Principles of Microeconomics), ECO 1312 (Inflation, Recession, and Unemployment), ECO 3301 (Price Theory), and MATH 1309 (Introduction to Calculus for Business and Social Sciences) or MATH 1337 (Calculus I).

1 Basic Derivative Rules for Univariate Functions

Univariate functions are functions of only one variable. For two such functions $f(x)$ and $g(x)$ and any two real numbers $c, n \in \mathbb{R}$, we have the following rules:

- **Constant rule**:
$$\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0.$$

- **Constant multiple rule**:
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\Big[c \cdot f(x)\Big] &= c \cdot \frac{\mathrm{d}f(x)}{\mathrm{d}x} \\ &\equiv c \cdot f'(x).\end{aligned}$$

- **Power rule**:
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right) = n \cdot x^{n-1}.$$

- **Sum rule**:
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\Big[f(x) + g(x)\Big] &= \frac{\mathrm{d}f(x)}{\mathrm{d}x} + \frac{\mathrm{d}g(x)}{\mathrm{d}x} \\ &\equiv f'(x) + g'(x).\end{aligned}$$

- **Difference rule**:
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\Big[f(x) - g(x)\Big] &= \frac{\mathrm{d}f(x)}{\mathrm{d}x} - \frac{\mathrm{d}g(x)}{\mathrm{d}x} \\ &\equiv f'(x) - g'(x).\end{aligned}$$

- **Product rule**:
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\Big[f(x) \cdot g(x)\Big] &= \frac{\mathrm{d}f(x)}{\mathrm{d}x} \cdot g(x) + f(x) \cdot \frac{\mathrm{d}g(x)}{\mathrm{d}x} \\ &\equiv f'(x) \cdot g(x) + f(x) \cdot g'(x).\end{aligned}$$

- **Quotient rule**:
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] &= \left(\frac{\mathrm{d}f(x)}{\mathrm{d}x} \cdot g(x) - f(x) \cdot \frac{\mathrm{d}g(x)}{\mathrm{d}x}\right)\Bigg/ g(x)^2 \\ &\equiv \frac{f'(x)g(x) - f(x) \cdot g'(x)}{g(x)^2}.\end{aligned}$$

- **Chain rule**:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) &= \frac{\mathrm{d}f\big(g(x)\big)}{\mathrm{d}g(x)} \cdot \frac{\mathrm{d}g(x)}{\mathrm{d}x} \\ &\equiv f'\big(g(x)\big) \cdot g'(x).\end{aligned}$$

Some Examples

- Suppose $f(x) = 3$. Then, by the constant rule,

$$f'(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}(3) = 0.$$

- Suppose $f(k) = Ak$, where $A > 0$. Then, by the constant multiple rule,

$$f'(k) \equiv \frac{\mathrm{d}}{\mathrm{d}k}(Ak) = A.$$

- Suppose $f(x) = x^3$. Then, by the power rule,

$$f'(x) \equiv \frac{\mathrm{d}}{\mathrm{d}k}(x^3) = 3 \cdot x^{3-1} = 3x^2.$$

- Suppose $f(x) = 2x^2$ and $g(x) = 3x$. Then, by the sum rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[f(x) + g(x)\big] = f'(x) + g'(x) = 4x + 3.$$

- Suppose $f(k) = Ak$ and $g(k) = wk$, where $A, w > 0$. Then, by the difference rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[f(x) - g(x)\big] = f'(x) - g'(x) = A - w.$$

- Suppose $f(x) = 2x^2$ and $g(x) = 3x$. Then, by the product rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\big[f(x) \cdot g(x)\big] &= f'(x)g(x) + f(x) \cdot g'(x) \\ &= 4x \cdot 3x + 2x^2 \cdot 3 = 18x^2.\end{aligned}$$

- Suppose $f(x) = 2x^2$ and $g(x) = 3x$. Then, by the quotient rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\Big[f(x) \cdot g(x)\Big] &= \frac{f'(x)g(x) - f(x) \cdot g'(x)}{g(x)^2} \\ &= \frac{4x \cdot 3x - 2x^2 \cdot 3}{(3x)^2} = \frac{2}{3}.\end{aligned}$$

- Suppose $f(x) = 2x^2$ and $g(x) = 3x$. Then, $f\big(g(x)\big) = f \circ g = 2\big(3x\big)^2 = 2 \cdot 9x^2 = 18x^2$. By the chain rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) &= f'\big(g(x)\big) \cdot g'(x) \\ &= 4g(x) \cdot 3 \\ &= 4(3x) \cdot 3 \\ &= 36x.\end{aligned}$$

2 Basic Derivative Rules for Multivariate Functions

Multivariate functions are functions of two or more variables. In this course, we generally work with two-variable functions of the form $z = f(x, y)$, where x and y are the two variables in question. One relevant example is a production function. We produce output Y according to some technology F that combines K and L. For example, the Cobb-Douglas production function:

$$Y = F(K, L) = AK^{\alpha}L^{\beta}, \qquad \alpha\beta \in (0, 1),\ A > 0.$$

2.1 Partial Derivatives

If a function is a multivariate function, we use the concept of partial differentiation to measure the effect of a change in one independent variable on the dependent variable, keeping the other variables constant. To apply the rules of calculus, we change only one independent variable and keep all other independent variables constant. In this way, we only look at the partial variation in the function instead of the total variation.

For instance, if a function is $f(x, y)$, we use *partial differentiation* with respect to x to measure the rate of change in $f(x, y)$ when only x changes and y remains constant. The partial derivative of f with respect to x is written as $\frac{\partial f}{\partial x}$ or simply as f_x. Similarly, the partial derivative of f with respect to y is written as $\frac{\partial f}{\partial y}$ or simply as f_y. In the case of a general production function $Y = F(K, L)$, the partial derivatives are denoted as:

$$\frac{\partial F(K, L)}{\partial K} \equiv F_K \qquad \text{and} \qquad \frac{\partial F(K, L)}{\partial L} \equiv F_L$$

We also refer to these derivatives as the marginal products of capital and labor (MPK and MPL), respectively. The marginal product of capital tells us how much output changes when we increase capital by one marginal unit while keeping labor constant, and the marginal product of labor tells us how much output changes when we inrease labor by one marginal unit while keeping capital constant.

To indicate that we are performing partial differentiations and not total differentiation, we use the sign $\frac{\partial f(x,y)}{\partial x}$ instead of $\frac{\mathrm{d}f(x)}{\mathrm{d}x}$. Partial differentiation with only one of the independent variables uses the same rules as the differentiation of one variable functions, except that while differentiation a function of several variables with respect to one independent variable, we keep all other independent variables constant.

Example I: Consider $f(x,y) = 3x^2 - 2y + 4$. The partial derivatives are:

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial(3x^2)}{\partial x} - \frac{\partial(2y)}{\partial x} + \frac{\partial(4)}{\partial x} = 6x - 0 + 0 = 6x,$$
$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial(3x^2)}{\partial y} - \frac{\partial(2y)}{\partial y} + \frac{\partial(4)}{\partial y} = 0 - 2 + 0 = 2.$$

When we compute $\frac{\partial f(x,y)}{\partial x}$, we treat y as constant; and when we compute $\frac{\partial f(x,y)}{\partial y}$, we treat x as constant.

Example II: Consider the profit function $\Pi(K,L) = PF(K,L) - WL - RK$. The partial derivatives are:

$$\frac{\partial \Pi(K,L)}{\partial K} = \frac{\partial\big(PF(K,L)\big)}{\partial K} - \frac{\partial(WL)}{\partial K} - \frac{\partial(RK)}{\partial K} = P\frac{\partial F(K,L)}{\partial K} - 0 - R = PF_K(K,L) - R,$$
$$\frac{\partial \Pi(K,L)}{\partial L} = \frac{\partial\big(PF(K,L)\big)}{\partial L} - \frac{\partial(WL)}{\partial L} - \frac{\partial(RK)}{\partial L} = P\frac{\partial F(K,L)}{\partial L} - W - 0 = PF_L(K,L) - W.$$

When we compute $\frac{\partial \Pi(K,L)}{\partial K}$, we treat L as constant; and when we compute $\frac{\partial F(K,L)}{\partial L}$, we treat K as constant.

Remark. As we can see in these examples, all rules of ordinary differentiation apply. You can think of partial derivatives as three step process:

1. **Use proper notation to indicate you are doing partial differentiation**; that is, use $\frac{\partial f(x,y)}{\partial x}$ instead of $\frac{\mathrm{d}f(x,y)}{\mathrm{d}x}$ to denote the derivative of f with respect to x.
2. **When taking the partial derivative of f with respect to variable x, treat any variable that is not x as a constant**.
3. **Apply the rules of ordinary differentiation when calculating partial derivatives**.

2.2 Second-Order Partial Derivatives

By taking the partial derivatives of the partial derivatives, we can compute higher-order derivatives. Given that a function $f(x,y)$ is twice-continuously differentiable, we can derive the following sets of second-order partial derivatives. First, we can derive the *direct* second-order partial derivatives:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \equiv \frac{\partial f_x}{\partial x} \qquad \text{and} \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} \equiv \frac{\partial f_y}{\partial y}.$$

Next, we can derive the cross-partial derivatives:

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial f_x}{\partial y} \qquad \text{and} \qquad f_{yx} = \frac{\partial^2 f}{\partial y \partial x} \equiv \frac{\partial f_y}{\partial x}.$$

Example: Consider production function $F(K,L) = AK^{\alpha}L^{\beta}$, where $\alpha, \beta \in (0,1)$ and $A > 0$. The second-order (direct) partial derivatives for this function are:

$$F_{KK} = \frac{\partial^2 F(K,L)}{\partial K^2} = \frac{\partial F_K}{\partial K} \equiv \frac{\partial(\alpha A K^{\alpha-1}L^{\beta})}{\partial K} = (\alpha - 1)\alpha A K^{\alpha-2}L^{\beta},$$
$$F_{LL} = \frac{\partial^2 F(K,L)}{\partial L^2} = \frac{\partial F_L}{\partial L} \equiv \frac{\partial(\beta A K^{\alpha}L^{\beta-1})}{\partial L} = (\beta - 1)\beta A K^{\alpha}L^{\beta-2}.$$

The cross-partial derivatives are:

$$F_{KL} = \frac{\partial^2 F(K,L)}{\partial K \partial L} = \frac{\partial F_K}{\partial L} \equiv \frac{\partial(\alpha A K^{\alpha-1}L^{\beta})}{\partial L} = \alpha\beta A K^{\alpha-1}L^{\beta-1} \equiv \alpha\beta\frac{Y}{KL},$$
$$F_{LK} = \frac{\partial^2 F(K,L)}{\partial L \partial K} = \frac{\partial F_L}{\partial K} \equiv \frac{\partial(\beta A K^{\alpha}L^{\beta-1})}{\partial K} = \alpha\beta A K^{\alpha-1}L^{\beta-1} \equiv \alpha\beta\frac{Y}{KL}.$$

As you can see, the two cross-partial derivatives of the Cobb-Douglas production are identical. This is because function F is symmetric in K and L. You can also verify that the second-order derivatives are negative, and that the cross partials are positive.

2.3 Using Partial Derivatives

For a twice-continuously multivariate function, the first-order partial derivatives are the marginal functions, and the second-order direct partial derivatives measure the slope of the corresponding marginal functions.

For example, if the function $f(x,y)$ is twice-continuously differentiable, the marginal functions are f_x and f_y, and the slopes of the marginal functions are given by f_{xx} and f_{yy}, respectively.

Marginal rate of substitution (MRS). For twice-continuously differentiable functions $f(x,y)$, we can compute partial derivatives as shown above. We can use these partial derivatives to measure the rate of change of the function with respect to x divided by the rate of change of the function with respect to y, which is f_x/f_y. If there exists level curves for the function $f(x,y)$, the ratio f_x/f_y is called the marginal rate of substitution.

Example: Consider production function $F(K,L) = AK^{\alpha}L^{\beta}$, where $\alpha, \beta \in (0,1)$ and $A > 0$. In this case, the marginal rate of substitution between capital and labor is:

$$\text{MRS}_{KL} := \frac{MPK}{MPL} \equiv \frac{F_K(K,L)}{F_L(K,L)} = \frac{\alpha}{\beta}\frac{L}{K}$$

We can compute the marginal rate of substitution between capital and labor at any point (K,L). For instance, at $(K,L) = (5,25)$ for $\alpha = \beta = 0.5$, we have

$$\text{MRS}_{KL} = \frac{\alpha}{\beta}\frac{L}{K} = \frac{1/2}{1/2} \cdot \frac{25}{5} = 5$$

2.4 Partial Derivative and Optimization

Partial derivatives can be used to optimize an objective function of several variables subject to a constraint or a set of constraints, given that the functions are differentiable. Mathematically, a constrained optimization problem with an equality constraint is:

$$\begin{aligned} &\max_{x,y} f(x,y) \\ &\text{s.t.} \quad g(x,y) = b. \end{aligned}$$

Here, $f(x,y)$ is the objective function, $g(x,y) = b$ is the equality constraint, and x, y are the variables we want to maximize.

We can find the optimal levels of x and y by forming the Lagrange function:

$$\mathcal{L}(x,y,\lambda) = f(x,y) = \lambda\Big(g(x,y) - b\Big),$$

where λ is the Lagrange multiplier.[1] To find the critical points we take the first-order partial derivatives of the Lagrangian with respect to x, y and λ and set each one of them to zero. We have,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= 0, \\ \frac{\partial \mathcal{L}}{\partial y} &= 0, \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= 0 \end{aligned}$$

By solving this simultaneous system of equations, we find the critical value of the function, if any exists.

[1] The idea of adding an artificial variable λ is to transform the constrained optimization problem into an unconstrained one for which we can use the first-order conditions outlined above.

Example (Profit maximization). The problem of the firm is to choose how much capital and labor to hire given their technology to maximize its profits taking prices P, W, R as given. Formally, we can write the problem of the firm as:

$$\max_{K,L\geq 0} \Pi = PY - WL - RK$$
$$\text{s.t.} \quad Y = F(K,L).$$

Here, Π denotes profits, which are given by revenues PY net of labor costs WL and capital costs RK, and $Y = F(K,L)$ is a technological constraint—the firm produces output Y according to $Y = F(K,L)$.

In this case, we can rewrite the problem by plugging in the production function into the profit function; that is,

$$\max_{K,L\geq 0} \Pi = PF(K,L) - WL - RK.$$

The first-order conditions for this problem give us a solution for the optimal levels of capital and labor. These conditions are:

$$\frac{\partial \Pi(K,L)}{\partial K} = PF_K(K,L) - R = 0 \qquad \Longrightarrow \qquad R = P \cdot F_K(K,L),$$
$$\frac{\partial \Pi(K,L)}{\partial L} = PF_L(K,L) - W = 0 \qquad \Longrightarrow \qquad W = P \cdot F_L(K,L).$$

These conditions say that when the firm optimally chooses its levels of capital and labor, factors of production receive their marginal products. For $Y = AK^{\alpha}L^{\beta}$, we have:

$$R = P \cdot \alpha \frac{Y}{K},$$
$$W = P \cdot (1-\alpha)\frac{Y}{L}.$$

The rental rates of factors are inversely related to their quantities (i.e., the larger the number of workers L, the lower the wage W these workers receive; and the larger the number of machines K, the cheaper it is to rent these machines.)