ME2708 Economic Growth

Lectures 5&6: Innovation-based Models

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Outline

Introduction I

- Inability of first-generation endogenous growth models to solve the caveats of exogenous growth theory
  - Contradict empirical evidence (capital-labor shares, convergence, growth sustainability and unintentional technological progress, ...)
- These facts motivate a second wave of endogenous growth models: innovation-based models
- Innovation-based models aim to describe advanced economies that try to push forward the technological frontier
- Innovation-based models pivot on the following ideas:
  - innovation is profit-driven (requires imperfect competition)
  - innovation results from non-rivalrous but partially excludable ideas (presence of IRS and IPRs)
  - innovation reacts to market size on both supply (population) and demand (which directs technological change)

Introduction II

- Innovation-based models fall into two categories:
  1. **Product-variety models**: innovation causes productivity growth by creating new, not necessarily improved, varieties of products (Romer, 1987, 1990; Jones, 1995)
  2. **Schumpeterian growth theory**: innovation leads to creative destruction and growth (Sergestrom et al., 1990; Aghion and Howitt, 1988, 1992)

- Models we cover in these lectures:
  - Product-variety models: Romer (1990) and Jones (1995)
  - Schumpeterian models: model inspired in Aghion and Howitt (1992) but adapted to Romer's framework

Common Characteristics of Product-variety Models

- Productivity growth comes from expanded varieties of intermediate products
  - Expansions are gradual: they take time and other resources
- Each new product has a one-time fixed cost (e.g. research costs)
  - Fixed costs lead to imperfect competition, which allows for positive profits (the reward to innovation!)
    - ⋆ Key difference with neoclassical growth: under perfect competition, output is exhausted in the renumeration of capital and labor (*Euler's equation*) and technological progress A is not compensated
- Technological knowledge as a list of blueprints (one per innovation)
- The search for new ideas motivated by profit opportunities

- Romer's main contribution is to endogenize technological change in an economy of profit-maximizing agents

- Romer's model has three sectors:
  1. **Final-goods sector** produces output with the capital goods purchased from intermediate-good firms
  2. **Intermediate-goods sector**: monopolists that manufacture, after having acquired exclusive rights for production from the research sector, *unique* capital goods and sell them to final-good firms
  3. **Research sector** discovers new ideas and then creates blueprints (instructions to develop new technologies) that sells to intermediate-good firms

- Large number of firms that combine capital (obtained from the intermediate-goods sector) and labor to produce a homogenous final good Y
- Aggregate production function is given by,

$$Y(t) = L_Y(t)^{1-\alpha} \sum_{i=1}^{A} x_i(t)^{\alpha}, \qquad 0 < \alpha < 1$$

  where L_Y is the number of people engaged in production of the final good, each x_i is a capital good and A is the total number of capital goods available at time t

- This production function exhibits *CRS*

- For analytical convenience, we replace summation with an integral in the production function,

$$Y(t) = L_Y(t)^{1-\alpha} \int_0^A x_i(t)^\alpha di$$

interpretation is unaffected, but now there is a *range* $[0, A]$ of capital goods available for production

- There is perfect competition in this sector: large number of price-taker firms

- We take the price of final good Y as *numeraire*, i.e. normalized to 1

- Final-good firms need to decide how much labor to hire and capital to employ in the production of Y
    - They do so by maximizing profits (or, equivalently, minimizing costs)

- Firms must solve the following problem:

$$\max_{L_Y, x_i} \underbrace{L_Y(t)^{1-\alpha} \int_0^A x_i(t)^\alpha di}_{Revenues} - \underbrace{w(t)L_Y(t) - \int_0^A p_i(t)x_i(t)di}_{Costs}$$

where p_i is the rental price of capital good i and w is the wage paid to final-good laborers

- FOC yield,

$$p_i(t) = \alpha L_Y(t)^{1-\alpha} x_i(t)^{\alpha-1} \quad (1)$$

$$w_Y(t) = (1-\alpha)\frac{Y(t)}{L_Y(t)} \quad (2)$$

Note that these conditions apply to each capital good i

The Romer Model V

Framework V: The Final-goods Sector

- The final-good sector resembles production in the Solow model:
  - Firms rent capital until the marginal product of each capital good equals its rental price
  - Firms hire labor until the marginal product of labor equals the wage

- Monopolists produce *unique* capital goods and sell them to final-good producers
  - Firms incur a fixed cost to purchase patents (exclusive production rights) from the research sector

- Simple production function: 1 unit of raw capital $\rightarrow$ 1 unit of capital good x_i

- Intermediate-goods' producers must solve the problem

$$\max_{x_i} \pi_i = p_i(x_i)x_i - rx_i$$

where x_i is the capital good and $p_i(x_i)$ is the inverse demand function of this capital good

- This economy admits representative agents so that FOC yields:

$$p^{'}(x)x + p(x) - r = 0$$

- Slightly rearranging previous equation and dividing both sides by p,

$$\frac{r}{p} = \underbrace{p^{'}(x)\frac{x}{p}}_{\text{Elasticity}} + 1$$

- Solving for p,

$$p = \left(\frac{1}{1 + \frac{p'(x)x}{p}}\right) r$$

- The elasticity of substitution can be calculated from eq. (1),

$$\frac{\partial p}{\partial x}\frac{x}{p} = \alpha - 1$$

- Plugging the elasticity $1-\alpha$ into the price equation we obtain,

$$p = \left(\frac{1}{\alpha}\right) r$$

- Each monopolist sells at the same price, p, and asks for a markup over marginal cost r

- Since inverse demand function (1) is also the same for all firms, each capital good is employed by final-good firms in the same amount

$$x_i = x$$

- Also all intermediate-goods firms earn the same profit (*tricky*!)

$$
\begin{aligned}
\pi &= p(x)x - rx \\
&= \frac{\alpha L_Y^{1-\alpha} \int_0^A x^\alpha}{A} - \frac{\alpha p \int_0^A x}{A} \\
&= \frac{\alpha L_Y^{1-\alpha} \int_0^A x^\alpha}{A} - \frac{\alpha^2 L_Y^{1-\alpha} \int_0^A x^\alpha}{A} \\
&= \frac{\alpha Y}{A} - \frac{\alpha^2 Y}{A} \\
&= (\alpha - \alpha^2)\frac{Y}{A} \\
&= \alpha(1-\alpha)\frac{Y}{A} \qquad (3)
\end{aligned}
$$

- We have obtained the following results for intermediate-goods firms:
  1. they charge the same price $p_i = p$
  2. thy use the same amount of each capital good $x_i = x$
  3. they earn the same profit π
- Result (2) is especially important since it allows us to obtain a more convenient aggregate (final-goods) production function
- Total capital in the economy is given by,

$$\int_0^A x_i(t)di = K(t)$$

- Making use of result (2),

$$K(t) = A(t)x(t) \qquad \Leftrightarrow \qquad x(t) = \frac{K(t)}{A(t)} \tag{4}$$

- Final-goods production function can now be expressed as,

$$Y(t) = L_Y(t)^{1-\alpha} A(t) x(t)^{\alpha}$$

- Substituting $x(t)$ for the expression derived in eq. (4),

$$\begin{aligned} Y(t) &= L_Y(t)^{1-\alpha} A(t) A(t)^{-\alpha} K(t)^{\alpha} \\ &= K(t)^{\alpha} (A(t) L_Y(t))^{1-\alpha} \end{aligned} \tag{5}$$

which coincides with the labor-augmenting aggregate production function used throughout

The Romer Model XII

Framework XII: The Research Sector

- Individuals can freely decide whether to have a career in research or not; if successful, they are rewarded from selling their ideas/blueprints
- Blueprints: instructions to produce new capital goods (e.g. MacBook, electric car, etc.)
- Blueprints or ideas accumulate to form the stock of knowledge
- The stock of knowledge evolves according to the following *key* equation (several versions!),

$$\dot{A}(t) = \overline{\theta} L_A(t) \tag{6}$$

  where $\overline{\theta}$ is the average rate of discovering new ideas and L_A is the amount of labor devoted to research
- The rate of discovering new ideas could be constant, depend on previous knowledge stocks and/or on the number of people devoted to research (*positively or negatively*), and/or on any other factors!

- Romer (1990) models the average rate of discovering ideas as an increasing function of the available technology:

$$\overline{\theta}(t) = \theta A(t) \tag{7}$$

  where θ is a constant, and the rate $\overline{\theta}$ is proportional to the stock of technology available at time t

- Main difference between Romer (1990) and Jones (1995) models in the formulation of the motion of ideas (*more later*!)

- Plugging eq. (7) into eq. (6) we get,

$$\dot{A}(t) = \theta A(t) L_A(t) \tag{8}$$

  which is linear in both A and L_A.

- Larger knowledge stocks and higher number of researchers results in more technological progress

- When an inventor designs a new blueprint, she receives a patent right
  - For simplicity, we assume that patents last forever
- Inventors sell patents to intermediate-good firms for price P_A and use this money for consumption/saving
- P_A is the present discounted value of the profits to be earned with this new design
  - Determined at an auction in which any intermediate-good firm can bid
  - If the auction starts at $P_A^{'} > P_A$, no bids (losses)
  - As long as $P_A^{'} < P_A$, firms always willing to bid higher (profit opport.)

- Motion of P_A determined by the method of arbitrage:
  - Suppose you have some money (P_A) that can either invest in the bank and earn interest r or use to purchase a patent for one period, earn the profits π of that period and then sell for whatever price is worth $\dot{P}_A$
  - In equilibrium these options are equally profitable (otherwise people would invest in the most profitable investment, driving its return down)

$$rP_A = \pi + \dot{P}_A$$

  - We can rewrite this equation as,

$$r = \frac{\pi}{P_A} + \frac{\dot{P}_A}{P_A}$$

  - Along a *BGP*, r is constant; implying that π / P_A is also constant and both π and P_A grow at the same rate, which is n (*more on this later*!)

- The price of a patent along a *BGP* is then,

$$P_A = \frac{\pi}{r - n} \tag{9}$$

The Romer Model XVI

Framework XVI: Other Assumptions

- The labor in this economy is *fully* employed, and only in two sectors,

$$L = L_Y + L_A$$

- We make the *behavioral* assumption that the relative allocation of labor between sectors is constant (*in Romer's original paper, utility maximization pins down this number*, more later!)

$$s_R = \frac{L_A(t)}{L(t)} \qquad \text{and} \qquad s_Y = 1 - s_R$$

- As in Solow, we assume that a constant fraction of output is invested in physical capital, $s_K \in (0, 1)$

- The economy starts with initial endowments of capital, labor and ideas, respectively: $K(0), L(0)$ and $A(0)$

The Romer Model XVII

Aggregate Production Function

- Aggregate production function for this economy is given by eq. (5),

$$\begin{aligned} Y(t) &= F[K(t), A(t)L_Y(t)] \\ &= K(t)^{\alpha}(A(t)L_Y(t))^{1-\alpha} \end{aligned}$$

  where $0 < \alpha < 1$ and variables are interpreted as usual

- This production functions presents CRS to K and L_Y:
  - satisfying neoclassical technology assumption $KA1$ only for given levels of technology A
  - violating $KA1$ when A is endogenized, i.e. treated as input. Then, Y presents IRS:

$$F[\lambda K, (\lambda A)(\lambda L_Y)] > \lambda F[K, AL], \quad \forall \lambda > 1$$

    doubling inputs more than doubles output!

The Romer Model XVIII

Key Accumulation Equations

- Both capital and labor exhibit the same behavior as in basic Solow,

$$\dot{K}(t) = s_K Y(t) - \delta K(t)$$
$$\dot{L}(t) = nL(t)$$

- The new *key* equation for the motion of technology/ideas was given by eq. (8),

$$\dot{A}(t) = \theta A(t) L_A(t)$$

- What is the per capita growth rate along a BGP in this model?
- Define output and capital in per capita terms,

$$\tilde{y}(t) \equiv \frac{Y(t)}{L(t)} \qquad \text{and} \qquad \tilde{k}(t) \equiv \frac{K(t)}{L(t)}$$

- Aggregate production function, eq. (5), can then be expressed in per capita terms as,

$$\begin{aligned}\tilde{y}(t) &= f(\tilde{k}(t), (1 - s_R)A(t)) \\ &= \tilde{k}(t)^{\alpha}((1 - s_R)A(t))^{1-\alpha}\end{aligned}$$

- Taking logs and differentiating *wrt* time,

$$g_{\tilde{y}} = \alpha g_{\tilde{k}} + (1 - \alpha) g_A \tag{10}$$

The Romer Model XX

Growth II

- Recall that capital, output, consumption, and population must grow at constant rates along a *BGP*
- From the capital per capita accumulation equation is clear that $\tilde{y}$ and $\tilde{k}$ must grow at the same rate for $\tilde{y}/\tilde{k}$ to be constant over time,

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = s_K \frac{\tilde{y}(t)}{\tilde{k}(t)} - (\delta + n)$$

- When $\tilde{y}$ and $\tilde{k}$ grow at the same rate, equation (10) is satisfied *iff*,

$$g_{\tilde{y}} = g_{\tilde{k}} = g_A$$

- Output per capita, the capital-labor ratio and the stock of ideas grow all at the same rate along a *BGP*
  - No technological progress $\Rightarrow$ no per capita growth!

- The growth rate of technological progress along a *BGP* is then of the utmost importance! How does technological progress occur?

- Recall, once again, eq. (10)

$$\dot{A}(t) = \theta A(t) L_A(t)$$

and note that linearity in $A(t)$ makes unbounded growth possible!

- Equation (10) can be rewritten as,

$$g_A \equiv \frac{\dot{A}(t)}{A(t)} = \theta L_A(t)$$

- There is technological progress and sustained growth even if the number of researchers is held constant!

The Romer Model XXIII

Growth IV

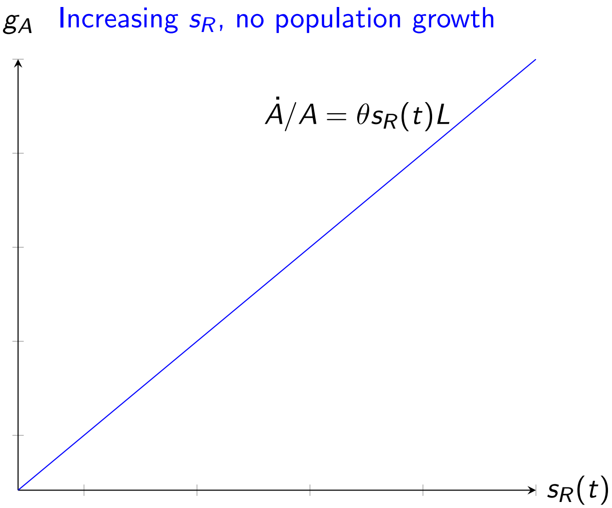
- Along a *BGP*, the number of researchers must be held constant so that g_A is also constant!

$$g_A \equiv \frac{\dot{A}(t)}{A(t)} = \theta L_A = \theta s_R L(t)$$

- An important implication of this model is that governments can positively affect *long-run growth rates* by policies that increase the number of researchers!
  - In contrast to the predictions of Jones' (1995) model
  - ...but it is a **conterfactual!**
    - ⋆ L_A in advanced economies has increased rapidly in the past few decades, yet growth rates haven't grown rapidly!
- Also, in contrast to Solow and neoclassical growth, an increasing population (*in this case of researchers*!) accelerates growth

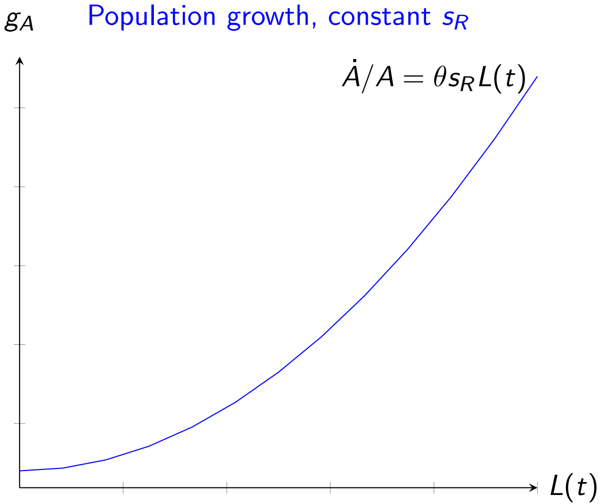
The Romer Model XXIV

Transition Path I: Increasing s_R with no population growth



The Romer Model XXV

Transition Path II: Population growth and constant s_R



The Romer Model XXVI

Analytical Solution I

- We solve the model now in terms of effective units of labor. We define,

$$\widehat{y}(t) \equiv \frac{Y(t)}{A(t)L(t)} \qquad \text{and} \qquad \widehat{k}(t) \equiv \frac{K(t)}{A(t)L(t)}$$

- Production function (5) can now be expressed as,

$$\begin{aligned}\widehat{y}(t) &= f(\widehat{k}(t), 1 - s_R) \\ &= \widehat{k}(t)^{\alpha}(1 - s_R)^{1-\alpha}\end{aligned}$$

- Taking logs and time derivatives of $\widehat{k}(t) \equiv K(t)/A(t)L(t)$,

$$\begin{aligned}\frac{\dot{\widehat{k}}(t)}{\widehat{k}(t)} &= \frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)} \\ &= \frac{s_K \widehat{y}(t)}{\widehat{k}(t)} - (\delta + g_A + n)\end{aligned}$$

The Romer Model XXVII

Analytical Solution II

- The capital accumulation equation in terms of effective units of labor,

$$\dot{\widehat{k}}(t) = s_K \widehat{y}(t) - (\delta + g_A + n)\widehat{k}(t)$$

- As usual, in the steady state $\widehat{k}$ must be constant,

$$\dot{\widehat{k}}(t) = 0 \Rightarrow s_K \widehat{y}^* = (\delta + g_A + n)\widehat{k}^*$$

- Plugging in the expression for $\widehat{y}^*$ we get,

$$s_K {\widehat{k}^*}^{\alpha} (1 - s_R)^{1-\alpha} = (\delta + g_A + n)\widehat{k}^*$$

- Solving for $\widehat{k}^*$,

$$\widehat{k}^* = \left(\frac{s_K}{\delta + g_A + n} \right)^{\frac{1}{1-\alpha}} \cdot (1 - s_R)$$

The Romer Model XXVIII

Analytical Solution III

- Plugging $\widehat{k}^*$ into $\widehat{y}^*$ we obtain,

$$\widehat{y}^* = \left(\frac{s_K}{\delta + g_A + n}\right)^{\frac{\alpha}{1-\alpha}} \cdot (1 - s_R)$$

- To better appreciate the role of technology we look at output per capita, $\tilde{y}^*$,

$$\begin{aligned}\tilde{y}^*(t) &= A(t)\widehat{y}^*(t) \\ &= A(0)e^{\theta s_R L t}\left(\frac{s_K}{\delta + g_A + n}\right)^{\frac{\alpha}{1-\alpha}} \cdot (1 - s_R)\end{aligned}$$

- Output per capita is higher for economies with higher investment rates. Two terms involve the share of labor devoted to research:
  - (1) the higher the share of people devoted to research, the more productive the technology available is
  - (2) the more researchers, the fewer workers producing output
  - (GE) a more populous economy achieves higher levels of output per capita

The Romer Model XXIX

Analytical Solution IV

- A very important part of the model is still unsolved: the allocation of labor between sectors
- We assume that individuals:
  - *freely* decide which sector to work on; and
  - indifferent between working in the final-goods sector or in research
- Once again, we use the concept of arbitrage
  - in equilibrium $w_Y = w_R$ (*otherwise laborers switch sector*)
- The wage of workers in the production of final-goods, w_Y, was given by eq. (2),

$$w_Y(t) = (1-\alpha)\frac{Y(t)}{L_Y(t)}$$

recall that these workers earn their marginal product

The Romer Model XXX

Analytical Solution V

- Researchers' wage depends on the value of the blueprints they create, P_A, and on their productivity (marginal product), $\overline{\theta}$, which individual researchers take as given,

$$w_R(t) = \overline{\theta} P_A(t)$$

- We can now apply the arbitrage condition and solve for s_R,

$$w_Y(t) = w_R(t) \qquad \Leftrightarrow \qquad (1-\alpha)\frac{Y(t)}{L_Y(t)} = \overline{\theta} P_A(t)$$

- Using the result $P_A(t) = \pi(t)/(r-n)$ from eq. (9),

$$(1-\alpha)\frac{Y(t)}{L_Y(t)} = \overline{\theta}\frac{\pi(t)}{r-n}$$

The Romer Model XXXI

Analytical Solution VI

- Using the result $\pi(t) = \alpha(1-\alpha)Y(t)/A(t)$ from eq. (3),

$$(1-\alpha)\frac{Y(t)}{L_Y(t)} = \bar{\theta}\frac{\alpha(1-\alpha)\frac{Y(t)}{A(t)}}{r-n}$$

- Some terms cancel and we can rewrite this equation as,

$$\frac{1}{L_Y(t)} = \frac{\bar{\theta}}{A(t)} \cdot \frac{\alpha}{r-n}$$

- We can now use eq. (6), $\dot{A}(t) = \bar{\theta}L_A(t)$, divide both sides by A and obtain $\bar{\theta}/A(t) = g_A/L_A(t)$. Substituting this expression in the above equation,

$$\frac{1}{L_Y(t)} = \frac{g_A}{L_A(t)} \cdot \frac{\alpha}{r-n}$$

- We can now rearrange the expression,

$$\frac{L_A(t)}{L_Y(t)} = \frac{\alpha g_A}{r - n} \qquad \Leftrightarrow \qquad \frac{s_R}{1 - s_R} = \frac{\alpha g_A}{r - n}$$

- Solving for s_R,

$$\begin{aligned} s_R(r - n) &= (1 - s_R)\alpha g_A \\ &= \alpha g_A - s_R \alpha g_A \end{aligned}$$

- Further rearranging,

$$s_R(r - n + \alpha g_A) = \alpha g_A$$

- Finally,

$$s_R = \frac{\alpha g_A}{r - n + \alpha g_A} = \frac{1}{1 + \frac{r-n}{\alpha g_A}}$$

The Romer Model XXXIII

Analytical Solution VIII

- How can we interpret this equation?

$$s_R = \frac{1}{1 + \frac{r-n}{\alpha g_A}}$$

  - the faster the economy grows (higher g_A), the higher the fraction of population doing research
  - the higher the discount rate $(r - n)$, the lower the fraction working in research

- One could solve for the interest rate r and plug it in, if desired, in the equation for s_R

- Imbedded in the interest rate r is a key characteristic of the model...

- Recall that,

$$p = \left(\frac{1}{\alpha}\right) r \qquad \Leftrightarrow \qquad r = \alpha p$$

- Now that we have the aggregate production function for this economy we can express p as,

$$\begin{aligned} p &= \alpha L_Y(t)^{1-\alpha} \left(\frac{K(t)}{A(t)}\right)^{\alpha-1} \\ &= \alpha \frac{Y(t)}{K(t)} \end{aligned}$$

- So that r can be expressed as,

$$r = \alpha p = \alpha^2 \frac{Y(t)}{K(t)}$$

- Notice that,

$$r = \alpha^2 \frac{Y(t)}{K(t)} < MPK = \alpha \frac{Y(t)}{K(t)}$$

- where r is the rental price of capital goods paid to intermediate-goods firms, and MPK comes from the aggregate production function

- In contrast to the Solow model (where all factors of production are paid their marginal products), the Romer model dictates that capital is paid less than its marginal product!
  - consequence of imperfect competition and IRS
  - difference between MPK and r compensates research efforts

The Romer Model XXXVI

Summarizing

- Product-variety, endogenous technological change model
  - most useful for analyzing advanced economies
- Three sectors in the model: final-goods, intermediate-goods and research
- The aggregate production function exhibits *IRS*
  - consequence of imperfect competition in the intermediate-goods sector
- There are no profits in the model:
  - although firms in the intermediate-goods sector sell capital at a price greater than marginal costs ($p = r/\alpha$), profits compensate research efforts
  - . . . and *all* rents compensate some input factor
- Output per capita grows, along a *BGP*, at the rate of technological progress, which governments can influence
  - by increasing the amount of labor devoted to research!

- Jones' model is basically Romer's model with two modifications concerning:
  1. the production of new ideas
  2. population growth

- We keep (*most of*) the framework from Romer's model and even the results from the *Analytical Solution* subsection
  - the modification regarding population growth has "no effect" in the **structure** of the analytical solutions since the model previously developed already accommodates, if any, population growth

- We now focus on the key modification concerning the production of new ideas and on how this in turn affects the growth results and the implications of the model

- Recall that, in the Romer model, the stock of knowledge evolves according to equation (6),

$$\dot{A}(t) = \overline{\theta} L_A(t)$$

where $\overline{\theta}$ is the average rate of discovering new ideas and L_A is the amount of labor devoted to research

- We said that there could be several versions for this model depending on the specification of $\overline{\theta}$

- Romer (1990) did model the average rate of discovering ideas as an increasing function of the available technology $A(t)$

$$\overline{\theta}(t) = \theta A(t)$$

- This linearity in $A(t)$, we said, is what makes unbounded growth possible

- We also saw that the implications caused by this modelling in the Romer model (namely, that governments could affect growth rates by increasing the number of researchers) are refuted by the available empirical evidence
  - the number of researchers in several advanced economies has been increasing but the growth rates have not!
- Jones (1995) sets to solve this controversy:
  - Knowledge stock still evolves according to eq. (6),

$$\dot{A}(t) = \overline{\theta} L_A(t)$$

  - ... but the rate of discovering new ideas, $\overline{\theta}$, is modelled in a more realistic manner
    - ⋆ non-linear differential equation depending on four factors (2 variables + 2 parameters): previous knowledge stock, number of researchers, knowledge spillovers and duplication efforts

- Jones models the average rate of discovering new ideas as,

$$\overline{\theta}(t) = \theta L_A(t)^{\lambda-1} A(t)^{\phi} \tag{11}$$

where $0 < \lambda \leq 1$ and $\phi < 1$ are the parameters reflecting duplication efforts in research and knowledge spillovers, respectively

- Plugging eq. (11) into eq. (6) we obtain,

$$\dot{A}(t) = \theta L_A(t)^{\lambda} A(t)^{\phi} \tag{12}$$

- The closer λ is to 0, the more "stepping on toes"

- How do knowledge stocks affect the production of ideas? For ϕ, there are three possibilities:

  1. **Positively** ($\phi > 0$): researchers stand "on the shoulders of giants"
  2. **Negatively** ($\phi < 0$): "finishing out" of ideas, i.e. as discoveries are made, posterior discoveries become increasingly difficult!
  3. **No relationship** ($\phi = 0$): new discoveries are independent from previous ones!

- Now we can also note that Romer's formulation is a very special case of Jones model (eq. 12) in which $\lambda = \phi = 1$

$$\dot{A}(t) = \theta L_A(t)^{\lambda} A(t)^{\phi}$$

- The similar procedure that we applied for Romer's model leads us to the result that along a BGP,

$$g_{\tilde{y}} = g_{\tilde{k}} = g_A$$

Output per capita, the capital-labor ratio and the stock of ideas grow all at the same rate

- Same implication: No technological progress $\Rightarrow$ no per capita growth!
- What is then the rate of technological progress along a BGP?
  - Different from Romer's given that we changed the specification of $\overline{\theta}$
- Let's see. . . !

- Recall eq. (12), which specifies the behavior of knowledge stocks,

$$\dot{A}(t) = \theta L_A(t)^{\lambda} A(t)^{\phi}$$

- Dividing both sides by $A(t)$,

$$\frac{\dot{A}(t)}{A(t)} = \theta L_A(t)^{\lambda} A(t)^{\phi-1}$$

$$= \theta \frac{L_A(t)^{\lambda}}{A(t)^{1-\phi}} \tag{13}$$

- Along a *BGP* growth rates are constant so that numerator and denominator must grow at the same rate!

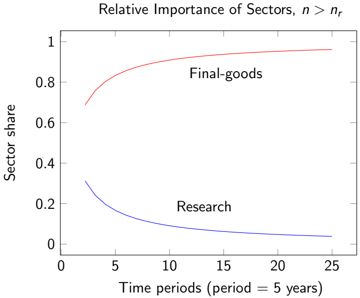
- Taking logs and time derivatives of this equation,

$$0 = \lambda \frac{\dot{L}_A(t)}{L_A(t)} - (1-\phi)\frac{\dot{A}(t)}{A(t)} \tag{14}$$

Jones VIII

Growth III

- Very important result *wrt* the growth rate of researchers must be deduced now: **the number of researchers grows at the same rate n as population does** (*why*?)
  - if it were to grow higher than n, there would eventually be more researchers than citizens (*a contradiction*!)
  - if it were to grow lower than n, the relative importance of the research sector would diminish over time (*contradicting the assumption of constant allocation of labor between sectors*) and $\lim_{t\to\infty} s_R = 0$



- Using $\dot{L_A}(t)/L_A(t) = n$, we can rewrite equation (14) as,

$$0 = \lambda n - (1 - \phi) g_A$$

- Solving for the growth rate of technological progress, g_A,

$$g_A = \frac{\lambda n}{1 - \phi} \qquad (15)$$

- Growth rate of technological progress determined by the parameters of the production of ideas and the growth rate of researchers/population
  - the less duplication efforts in research (higher λ), the higher the growth rate
  - the faster researchers/population grows (higher n), the higher the growth rate
  - the more knowledge spillovers (higher ϕ), the higher the growth rate

- **Crucial implication**: no researchers/population growth, no growth!

- Government intervention:

  "*The growth rate is determined by parameters that are typically viewed as invariant to policy manipulation*" − C. Jones

  - In contrast to Romer's model, and in line with Solow, subsidies to R&D or capital accumulation *only* have level effects; affecting the growth rate during the transition path
  - Although the model eliminates scale effects (in the long-run), scale effects are still important in the transition path (see eq. 13) which may be very long!

- Comparative statistics in Jones' may not be very intuitive as several parameters' interactions must be taken into account

$$\frac{\dot{A}(t)}{A(t)} = \theta \frac{L_A(t)^{\lambda}}{A(t)^{1-\phi}}$$

  - the higher L_A, the higher g_A
  - the higher A, the lower g_A for $0 < \phi < 1$
  - the less "stepping on toes" ($\lambda \to 1$), the higher g_A for given L_A
  - the more standing on the "shoulders of giants" (higher ϕ), the higher g_A for given A

Jones XII

Comparative Statistics II

- The book, for simplicity, only considers the case of $\lambda = 1, \phi = 0$, so that

$$\frac{\dot{A}(t)}{A(t)} = \theta \frac{L_A(t)}{A(t)} = \theta \frac{s_R L(t)}{A(t)}$$

- Along a *BGP* growth rates are constant. Taking logs and time derivatives of this last expression:

$$0 = n - g_A \quad \Rightarrow \quad g_A = n$$

  so that the growth rate of technology is equal to the population growth rate!

- What happens to technological progress (the growth rate of the economy) if the government manages to increase the share of the research sector permanently?
  - Next figure...!

- At the time of the policy, the increase in s_R raises g_A above n because researchers produce more ideas; over time, however, this ratio declines (because of higher A)

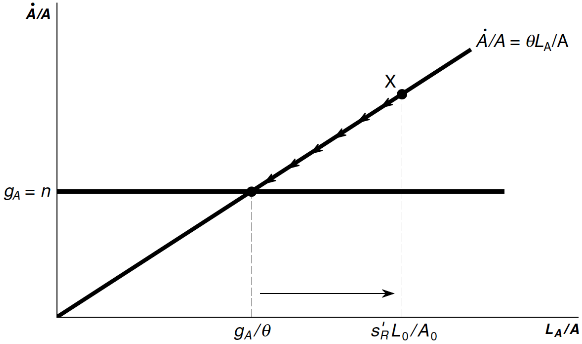


Figure: Permanent increase in the share of the research research sector

- Permanent increases in s_R raise g_A temporarily, but not in the long-run (the economy returns to a BGP where $g_A = n$)

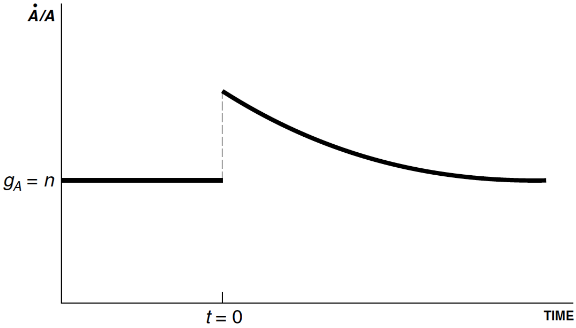


Figure: No long-run growth effects

- Level of technology (also of income) is permanently higher as result of the increase in s_R

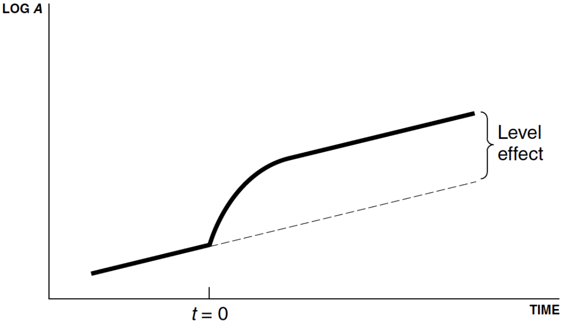


Figure: Level effects

Taking Stock I

- Romer and Jones' models are very similar analytically
  - key differences are found in: 1) the production of ideas (linear vs. non-linear differential equation in A and L_A); and 2) assumptions regarding population growth

- Slight modifications have profound implications. Along a BGP,

$$g_A = \theta s_R L \quad \text{(Romer)} \qquad \text{vs.} \qquad g_A = \frac{\lambda n}{1-\phi} \quad \text{(Jones)}$$

  - government's role is way less clear in Jones' than in Romer's

  - scale effects play a crucial role in Romer's model all along but are less important in Jones' (no long-run scale effects but rather temporary, i.e. along the transition phase)

Taking Stock II

Table: Models comparison: Romer vs. Jones

Romer	Jones
$g_Y = \alpha g_K + (1-\alpha) g_A$	$g_Y = \alpha g_K + (1-\alpha)(g_A + n)$
$g_K = s_K \left(\frac{Y}{K}\right) - \delta \;\; \Rightarrow \;\; g_Y = g_K = g$ (along a BGP)	
$g = g_A$	$g = g_A + n$
$g_A = \theta L_A$	$g_A = \frac{\lambda n}{1-\phi}$
$g = \theta L_A$	$g = \frac{\lambda n}{1-\phi} + n$

Schumpeterian Models I

- Product-variety models see technological progress as the result of ever-increasing varieties of capital goods
  - shortcoming: no obsolescence; each variety, once invented, lives forever
  - counterfactual: computers invented in 1990 should be used as much as the cutting-edge 2017 MacBook Pro
- As a response to the shortcomings of these models, models of Schumpeterian growth were developed
  - growth is generated by quality-improving innovations (there is obsolescence and innovations may be replaced)
  - creative destruction: innovation drives growth by creating new technologies that render obsolete old ones
  - important step forward since it brings insights of industrial organization: important role to exit and turnover of firms and workers

Framework I

- As in Romer and Jones, there are three sectors:
  1. Final-goods
  2. Intermediate-goods
  3. Research

- *Difference I*: there is a single monopolistic, intermediate-goods firm that owns the patent of available technology (in contrast to Romer and Jones' where there are several intermediate-good firms, each producing capital good x_i)

- *Difference II*: the research sector only looks for a patent at a time, the quality-improving innovation (in contrast to Romer and Jones' where researchers look for several patents)
  - *Creative destruction*: the intermediate-goods firm that purchases the patent for the new technology replaces the incumbent firm

- The production function for the final goods sector is,

$$Y(t) = x_i(t)^{\alpha}(A_i L_Y(t))^{1-\alpha}$$

importantly, i indexes ideas (note x is indexed by t but A is not); and x_i can be thought of as the quantity of i-related capital

- Large number of firms that always purchase the latest version of the capital good as it gives them the highest level of technology
- The problem of the these firms is to decide how much capital, x_i, to use and labor, L_Y, to hire in order to maximize profits, i.e.,

$$\max_{x_i, L_Y} \ \pi \equiv x_i(t)^{\alpha}(A_i L_Y(t))^{1-\alpha} - w_Y(t)L_Y(t) - p(t)x_i(t)$$

- where w is the rental price (wage) of labor and p is the rental price of capital good x_i

- Applying *FOC*:

$$\frac{\partial \pi}{\partial x_i(t)} = 0 \quad \Rightarrow \quad p(t) = \alpha x_i^{\alpha-1}(A_i L_Y(t))^{1-\alpha} \tag{16}$$

and

$$\frac{\partial \pi}{\partial L_Y(t)} = 0 \quad \Rightarrow \quad w_Y(t) = (1-\alpha) x_i^{\alpha} A_i^{1-\alpha} L_Y(t)^{-\alpha} = (1-\alpha)\frac{Y(t)}{L_Y(t)}$$

- Firms purchase capital goods and hire labor until their marginal products are equal to their rental prices $p(t)$ and $w_Y(t)$, respectively

Framework IV

The Intermediate-goods Sector I

- There is one single monopolist firm producing the only version of the i-related capital good
  - monopoly power obtained from patent rights
- Capital goods produced in a simple manner:
  - unit of raw capital $\rightarrow$ one unit of capital good
- The problem of this firm is,

$$\max_{x_i} \; \pi = p(x_i(t))x_i(t) - r(t)x_i(t)$$

where $p(x_i(t))$ is the inverse demand function of capital good x_i

Framework V

The Intermediate-goods Sector II

- Profit maximization yields,

$$p^{'}(x_i(t))x_i(t) + p(x_i) - r = 0$$

which can be rewritten as,

$$p = \left(\frac{1}{1 + \frac{p^{'}(x_i(t))x_i(t)}{p(t)}} \right) r$$

calculating the elasticity of demand from eq. (16), which is $\alpha - 1$,

$$p(t) = \left(\frac{1}{\alpha} \right) r$$

- so that the intermediate firm charges a constant markup, α, over marginal costs (*as in Romer and Jones'*)

Framework VI

The Intermediate-goods Sector III

- The firm's profits are given by,

$$\pi(t) = \alpha(1-\alpha)Y(t)$$

although similar to Romer and Jones', profits are not divided by the number of firms A because there is one single monopolist!

- Since there is only one single monopolist, all capital in the economy must be used to produce the latest intermediate good so that $x_i = K$ and we have an aggregate production function,

$$Y(t) = F[K(t), A_i L_Y(t)] = K(t)^{\alpha}(A_i L_Y(t))^{1-\alpha} \tag{17}$$

- Aggregate production function similar to Solow's or Romer-Jones',

$$Y(t) = F[K(t), A_i L_Y(t)] = K(t)^{\alpha}(A_i L_Y(t))^{1-\alpha}$$

where $0 < \alpha < 1$ and technology is indexed by i rather than by time t

  - $i = \{1, 2, \ldots\}$ can be thought of as the generation of technology that is available at a given point in time
  - in terms of warfare, generation 1 could be "lances", generation 2 could be "swords", generation 3 "rifles", generation 4 "automatic weapons"
    - ⋆ similar examples for transportation, production, computers, and so on
  - a given generation of technology may be available for one or more time periods
  - the elapsed time between technology generations needs not to be homogenous, i.e. technology's generation 1 may be functional for shorter, equal or longer time than technology's generation $i \neq 1 \in \mathbb{N}$

- All people working in research work for version $i+1$ of technology (in contrast to Romer and Jones where there may be several varieties)
  - ⋆ if successful, a more-productive version of technology $i+1$ is created

- Two factors *directly* influence the growth in A:
  1. **the average probability of innovation**, $\overline{\mu}$: depends on research efforts and is affected by externalities (stepping on toes, λ, and standing on the shoulders of giants, ϕ)

$$\overline{\mu} = \theta \frac{L_A(t)^{\lambda-1}}{A_i^{1-\phi}}$$

The economy's probability of innovation is, by the $L.L.N.$,

$$P(\text{innovation}) = \overline{\mu} L_A(t) = \theta \frac{L_A(t)^{\lambda}}{A_i^{1-\phi}}$$

so that it is higher: the higher the number of researchers, the less stepping on toes, the lower the level of technology and the more standing on shoulders (for given $0 < \phi < 1$)

Framework IX

The Research Sector II

2. **the size of innovation**, $\gamma > 1$: captures the size of innovation (or the productivity increase when innovation occurs)
   - ⋆ technology becomes more productive from one generation A_i to the next $A_{i+1} = \gamma A_i$

- Successful innovation provides the inventor with a patent to sell to the (new) monopolistic intermediate-goods firm, which replaces the previous one

- Value of the patents calculated, once again, by the *arbitrage method*,

$$r(t)P_A(t) = \pi(t) + \dot{P}_A(t) - \overline{\mu}L_A(t)P_A(t)$$

- The main difference with Romer and Jones is that patents lose *all* their value when a new technology arrives
  - ▶ P(losing patent value)$=$ P(innovation) $= \overline{\mu}L_A(t)$

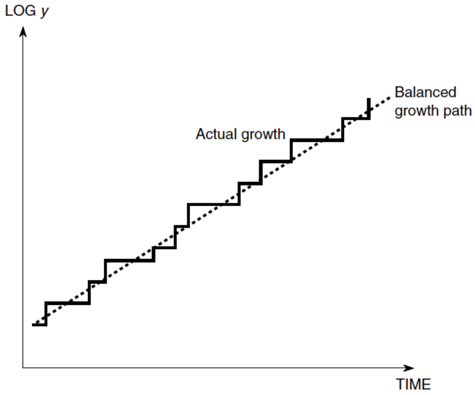
- Dividing both sides of the previous equation by $P_A(t)$,

$$r(t) = \frac{\pi(t)}{P_A(t)} + \frac{\dot{P}_A(t)}{P_A(t)} - \overline{\mu} L_A(t)$$

- Along a BGP, $r(t)$ and the ratio $\pi(t)/P_A(t)$ must be constant, i.e. $r(t) = r$, and both π and P_A grow at the same constant rate
  - we know that this rate is proportional to the growth rate of output (by looking at the intermediate-goods sector $\pi(t) = \alpha(1-\alpha)Y(t)$)
  - ...to solve this part of the model we need to understand growth first!
  - the good thing is that the model is tractable enough to do this now

Growth I

- There are periods of growth (jumps) and periods of stagnation because growth results from innovation (*a risky business*!)
  - only meaningful to talk about growth over long periods of time



Growth II

- From production function (17),

$$Y(t) = F[K(t), A_i L_Y(t)] = K(t)^\alpha (A_i L_Y(t))^{1-\alpha}$$

we can take logs and differentiate *wrt* time,

$$g_Y = \alpha g_K + (1-\alpha)(g_A + n)$$

- Along a *BGP*, aggregate output must grow at the same rate as aggregate capital ($g_Y = g_K = g$) so that,

$$g = g_A + n \quad \Leftrightarrow \quad g_A = g - n$$

- Solving for the growth rate of technology pins down the growth of other quantities

Growth III

- We introduce the *expectation operator* to reflect that technological progress is uncertain: it depends on the probability of innovation and the size of innovation

$$\mathbb{E}\left[\frac{\dot{A}_i}{A_i}\right] = \gamma\bar{\mu}L_A = \gamma\theta\frac{L_A^\lambda}{A_i^{1-\phi}} \tag{18}$$

- This corresponds to the *expected* average growth rate over long periods of time, so that we can rewrite,

$$g_A = \gamma\theta\frac{L_A^\lambda}{A_i^{1-\phi}}$$

- Taking logs, differentiating *wrt* time and imposing the *BGP* condition, we can write this equation in a more convenient manner,

$$g_A = \frac{\lambda n}{1-\phi}$$

which coincides with the growth rate of technology in Jones' model!

- Now we know that output grows at

$$g \equiv g_Y = g_A + n$$

and that both π and P_A grow at this rate because, along a BGP the ratio $\pi(t)/P_A(t)$ is constant and π is proportional is proportional to Y

- Using this fact, defining $\mu = \overline{\mu} L_A(t)$ and imposing the BGP condition for r,

$$r = \frac{\pi(t)}{P_A(t)} + \underbrace{\frac{\dot{P}_A(t)}{P_A(t)}}_{\gamma\mu+n} - \mu = 0$$

- We can solve for the value of a patent, P_A,

$$r = \frac{\pi(t)}{P_A(t)} + \gamma\mu + n - \mu \quad \Leftrightarrow \quad r - n + \mu(1-\gamma) = \frac{\pi(t)}{P_A(t)}$$

- Solving for P_A,

$$P_A(t) = \frac{\pi(t)}{r - n + \mu(1-\gamma)} \tag{19}$$

- The arbitrage equation implies that the price of a patent along a *BGP* depends on:
  - positively on profits, π, population growth, n, and size of innovation, γ
  - negatively on rental price of capital, r, and probability of innovation, μ
- The difference with Romer and Jones' models is the introduction of probability of innovation (μ) and size of innovation (γ) to reflect that:
  - new innovations bring about creative destruction

Allocation of Labor I

- Only the allocation of labor across sectors remains to be solved
- Once again, this is solved by using the arbitrage method:
  - individuals *freely* decide in which sector to work on
  - in equilibrium, $w_Y(t) = \mathbb{E}[w_R(t)]$ (*otherwise workers switch sectors*)
  - individuals working in final-goods production earn,

$$w_Y(t) = (1-\alpha)\frac{Y(t)}{L_Y(t)}$$

  - and individuals working in the research sector earn,

$$\mathbb{E}[w_R(t)] = \overline{\mu} P_A(t)$$

the *expectation operator* highlights that researchers earn the expected wage if they team-up and work in a large scale-lab (otherwise, salary is either $P_A(t)$, when successful, or 0 when unsuccessful)

Allocation of Labor II

- Imposing the equilibrium condition, $w_Y(t) = \mathbb{E}[w_R(t)]$, we get,

$$\overline{\mu} P_A(t) = (1-\alpha)\frac{Y(t)}{L_Y(t)}$$

- Substituting in $P_A(t)$ (eq. 19) and $\pi(t)$,

$$\frac{\overline{\mu}}{r - n + \mu(1-\gamma)}\alpha(1-\alpha)Y(t) = (1-\alpha)\frac{Y(t)}{L_Y(t)}$$

- Canceling common terms and substituting in $\overline{\mu} = \mu L_A(t)$,

$$\frac{\alpha\mu}{r - n + \mu(1-\gamma)} = \frac{L_A(t)}{L_Y(t)}$$

Allocation of Labor III

- Using $L_A(t)/L_Y(t) = s_R/(1 - s_R)$,

$$\frac{\alpha\mu}{r - n + \mu(1 - \gamma)} = \frac{s_R}{1 - s_R}$$

- Rearranging,

$$\begin{aligned} & (1 - s_R)\alpha\mu = s_R[r - n + \mu(1 - \gamma)] \\ \Leftrightarrow\ & \alpha\mu - s_R\alpha\mu = s_R[r - n + \mu(1 - \gamma)] \\ \Leftrightarrow\ & \alpha\mu = s_R[\alpha\mu + (r - n + \mu(1 - \gamma))] \end{aligned}$$

- Solving for s_R,

$$s_R = \frac{\alpha\mu}{\alpha\mu + r - n + \mu(1 - \gamma)} = \frac{1}{1 + \frac{r - n + \mu(1 - \gamma)}{\alpha\mu}}$$

Allocation of Labor IV

- The share of labor working in the research sector,

$$s_R = \frac{1}{1 + \frac{r - n + \mu(1-\gamma)}{\alpha\mu}}$$

depends:

- negatively on the discount rate to profits, $r - n$;
- and on the probability of innovation, μ,
  (1) negatively, $\mu(1-\gamma)$: as the chance of innovation increases, the value of a patent declines

  (2) positively, $\alpha\mu$: as the chance of innovation increases, individuals are more likely to engage in research

(GE) **net effect is positive**: if μ increases, so does s_R

Optimal R&D I

- Is the share of the population working in research optimal?
  - **No** (Romer, Jones, Schumpeterian Growth)

- Why not?
  - *Romer*: a higher s_R always leads to higher growth
  - *Jones and Schumpeterian growth*: s_R is too low!
    1. positive externalities not internalized (individuals do not take into account that their work set firmer ground for future researchers) $\Rightarrow$ too little research
    2. negative externalities (duplication efforts) $\Rightarrow$ research not "as efficient as desired"
    3. consumer-surplus effect: societal gain $>$ private gain $\Rightarrow$ too little research

Optimal R&D II

- Even with modern patent systems, the market tends to provide too little research
- Economics of innovation, in contrast to classical economics, suggest that it is crucial that firms have monopoly power ($p > MC$)
  - it is this wedge what provides incentives to innovate

Summary I

Innovation-based models:

- emerge to solve the difficulties of exogenous growth and the controversies of first-generation models of endogenous growth
- pivot around: i) agents are profit-driven (requires imperfect competition), ii) presence of IRS and IPRs, and iii) market size effects
- fall into 2 categories: product-variety models and Schumpeterian growth theory
- most often have three sectors: final-goods, intermediate-goods and research

Summary II

- Of the models studied, only Romer's governmental intervention has long-run growth effects
  - in Jones' and Schumpeterian growth, government policy may have *level* effects (in contrast to Romer's, which has growth effects)

- Main differences across models when modeling technological progress (Romer vs. Jones vs. Schumpeterian), which has effects on:
  - patent value, P_A
  - relative importance of research sector, s_R

- Analytical solutions for capital- and income- per capita are similar! Richer countries are those with:
  - higher investment rates
  - higher technology levels
  - higher relative importance of the research sector
  - higher technological progress
  - higher population growth

Summary III

- An important contribution of Schumpeterian growth is that it introduces insights from industrial organization
  - creative destruction brings in firm dynamism and entrepreneurial behavior
- As before, technological progress is the engine of growth
  - now is not "*manna from heaven*" but the result of purposeful activity
- Technological progress is, to a great extent, pinned down by population growth (*also by externalities*)
- In general, the allocation of labor to the research sector is *not* optimal: positive- (societal gain $>$ private gain) and negative- (duplication efforts) externalities
- Innovation-based models have a much better fit with the empirical evidence to date
  - one of the most satisfactory models in this respect is a mix of innovation-based models that allows both: introduction of new product varieties and substitution of old ones (beyond the scope of this course)

Thank you for your attention!