ME2708 Economic Growth

Lectures 2&3: The Solow Growth Model

Luis Perez

KTH Royal Institute of Technology

March 22, 2018

Outline

Introduction

- **Goal**: to develop a simple framework to think about the *proximate causes* (technological progress, physical- and human- capital) of economic growth and the factors that could *potentially* explain cross-country income differences
- Solow-Swan model, most often referred to as Solow Growth Model (Solow, 1956; Swan, 1956): game changer!
- Before the Solow model, the most common approach to economic growth built on the Harrod-Domar model (Harrod, 1939; Domar, 1946)
- However, Solow demonstrated that the Harrod-Domar model was problematic!
- At the center of the Solow model is the *neoclassical* (aggregate) production function, thus connecting with microeconomics and bridging theory with empirics!

Environment and Assumptions

Households and Production I

- Closed, one-good economy
- **Discrete time** running to an infinite horizon ($t = 0, 1, \ldots$) or **continuous time** ($t \in \mathbb{R}_+$)
- The economy is inhabited by a large number of households (e.g. $i \in [0, 1]$) that will not be optimizing!
  - Main difference between Solow and Neoclassical growth models
- To simplify analysis, assume that *all* households are identical so that the economy admits a *representative hoursehold*!
- Assume that households save a constant exogenous fraction $s \in (0, 1)$ of their disposable income
  - Same assumption as in basic Keynesian models but at odds with reality (e.g. government plans a tax increase in $t + 1 \ldots$ what about savings?)

Environment and Assumptions
Households and Production II

- Assume that *all* firms employ the same production function, thus the economy admits a *representative firm*!
- Aggregate production function given by

$$Y(t) = F[K(t), L(t), A(t)] \tag{1}$$

- $Y(t)$ is the *unique* final good (e.g. whatever you prefer!), $K(t)$ is the capital stock (e.g. machines), $L(t)$ is total employment (e.g. total hours or employees), and $A(t)$ is TFP (e.g. technology)
- $A(t)$ thought of as the "efficiency" shifter of the production function
- Technology, assumed **free**: publicly available, *non-excludable* (others can use it) and *non-rival* (others' use does not prevent own use!)

KA1: Continuity, Differentiability, Positive and Diminishing Marginal Products, and Constant Returns to Scale

The production function $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ is twice continuously differentiable in K and L, and satisfies

$$F_K(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, \qquad F_L(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial L} > 0$$

$$F_{KK}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \qquad F_{LL}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0$$

Moreover, F exhibits constant returns to scale in K and L, i.e. linearly homogenous (homogenous of degree 1) in these two variables!

$$F(\lambda K, \lambda L) = \lambda Y, \;\; \forall \lambda > 1 \Rightarrow F \text{ is homogeneous of degree 1 in } K \text{ and } L$$

KA2: Inada (1961) Conditions

F satisfies

$$\lim_{K\to 0} F_K(K, L, A) = \infty \quad \text{and} \quad \lim_{K\to\infty} F_K(K, L, A) = 0, \forall L > 0, A$$

$$\lim_{L\to 0} F_L(K, L, A) = \infty \quad \text{and} \quad \lim_{L\to\infty} F_L(K, L, A) = 0, \forall K > 0, A$$

Moreover, $F(0, L, A) = 0$ for all L and A.

- Inada (boundary) conditions ensure the existence of *interior equilibria*
- Assumptions $KA1$ and $KA2$, taken together, constitute what's called "**neoclassical technology assumptions**"

- Inada conditions, $KA2$, imply that first units of capital (labor) are highly productive but when these are sufficiently abundant their marginal products are close to zero!

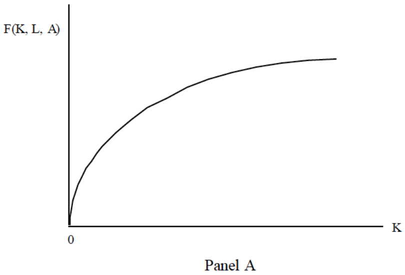


Figure: Production function A satisfies Inada conditions whilst B does not

- Assume that markets are *competitive*, i.e. agents are price-takers and prices clear markets!
- Households own all labor, which they supply inelastically (*regardless of its rental price*!)
- Endowment of labor is equal to population size, $\overline{L}(t)$
- Labor market clearing condition given by

$$L(t) = \overline{L}(t), \quad \forall t \in \mathbb{R}_+$$

- Rental price of labor or, most-commonly used, *wage rate* is $w(t)$. Then it must be satisfied,

$$L(t) \leq \overline{L}(t), w(t) \geq 0 \quad \text{and} \quad (L(t) - \overline{L}(t))w(t) = 0$$

- Also assume that households own *all* capital and rent it to firms at rental price of capital $R(t)$
- Capital market clearing condition given by

$$K(t) = \overline{K}(t)$$

- We take initial capital endowments $K(0) \geq 0$ as given!
- Prices, $P(t)$, are abstracted from the model: normalized to 1 in all periods (final output is the *numeraire*!)
- Capital depreciates at exponential rate $\delta \in (0, 1)$: 1 unit of capital today is $1 - \delta$ units tomorrow!
- Loss of capital affects interest rates (return to savings) of households

$$r(t) = R(t) - \delta$$

- The problem of the (*representative*) firm is to maximize profits

$$\max_{L(t)\geq 0, K(t)\geq 0} \pi \equiv \underbrace{F[K(t), L(t), A(t)]}_{\text{Revenues}} - \underbrace{w(t)L(t) - R(t)K(t)}_{\text{costs}}$$

- Equivalently, the problem can be formulated as to minimize costs
  $\Rightarrow$ Dual problem!

- The price of the final good $P(t)$ has been normalized to 1 (*the numeraire!*) and multiplies the F term

- The program already imposes *competitive* markets: firms take factor prices $w(t)$ and $R(t)$ as given!

Environment and Assumptions

Firm Optimization II

- Since F is differentiable, FOC yield:

$$R(t) = F_K[K(t), L(t), A(t)] \tag{2}$$

and

$$w(t) = F_L[K(t), L(t), A(t)] \tag{3}$$

- Solving for $K(t)$ and $L(t)$ we can derive the capital and labor demands of firms at rental prices $R(t)$ and $w(t)$

Proposition 1 (Simplified version of Euler's Theorem)

Suppose KA1 holds. Then in the equilibrium of the Solow model, firms make no profits and the following equation is satisfied

$$Y(t) = w(t)L(t) + R(t)K(t)$$

Fundamental Law of Motion I

- K depreciates exponentially at rate δ,

$$K(t+1) = (1-\delta)K(t) + I(t) \tag{4}$$

where $I(t)$ denotes investment at time t

- In a closed economy, abstracting from the government,

$$Y(t) = C(t) + I(t) \tag{5}$$

where C(t) denotes consumption. What is not consumed is invested,

$$S(t) = I(t) = Y(t) - C(t)$$

- Equation (4) can then be rewritten as,

$$\begin{aligned} K(t+1) &= Y(t) + (1-\delta)K(t) - C(t) \\ &= F[K(t), L(t), A(t)] + (1-\delta)K(t) - C(t) \end{aligned}$$

Fundamental Law of Motion II

- **Solow's behavioral rule**: households save a constant fraction $s \in (0,1)$ of their income,

$$S(t) = I(t) = sY(t) \tag{6}$$

$$C(t) = (1-s)Y(t) \tag{7}$$

- Equation (4) could also be rewritten as,

$$\begin{aligned} K(t+1) &= (1-\delta)K(t) + sY(t) \\ &= (1-\delta)K(t) + sF[K(t), L(t), A(t)] \end{aligned} \tag{8}$$

- This (nonlinear difference) equation is the *fundamental law of motion* in the Solow growth model!

- Equation (8) together with the laws of motion of $L(t)$ and $A(t)$ describe the equilibrium in the Solow growth model

Definition of Equilibrium

- The Solow model combines features of both Keynesian models (e.g. behavioral rules) and modern macroeconomic approaches (e.g. firm maximization and market clearing)

Definition of Equilibrium

In the basic Solow model for a given sequence of $\{L(t), A(t)\}_{t=0}^{\infty}$ and an initial capital stock $K(0)$, an equilibrium path is a sequence of capital stocks, output levels, consumption levels, rental rates and wages $\{K(t), Y(t), C(t), R(t), w(t)\}_{t=0}^{\infty}$ such that $K(t)$ satisfies equation (8), $Y(t)$ is given by equation (1), $C(t)$ is given by equation (7), and $R(t)$ and $w(t)$ are given by equations (2) and (3), respectively.

- An equilibrium in this context is understood as an "entire path" of allocations and prices, not a static object!

Equilibrium without Pop. Growth and Tech. Progress I

- Assume that
  1. No population growth: $L(t) = L$, at some level $L > 0$
  2. No technological progress: $A(t) = A$

- Define the capital-labor ratio of the economy as

$$\tilde{k}(t) \equiv \frac{K(t)}{L} \tag{9}$$

- We can then express output per capita, $\tilde{y}(t) \equiv Y(t)/L$, as

$$\begin{aligned} \tilde{y}(t) &= F\left[\frac{K(t)}{L}, 1, A\right] \\ &= f(\tilde{k}(t), A) \end{aligned} \tag{10}$$

  Since there is no technological progress, A is constant and can be omitted from the equation if normalized to 1,

$$\tilde{y} = f(\tilde{k}(t))$$

- Consider the following Cobb-Douglas production function,

$$\begin{aligned} Y(t) &= F[K(t), L(t), A(t)] \\ &= AK(t)^{\alpha} L(t)^{1-\alpha}, \quad 0 < \alpha < 1 \end{aligned}$$

- It satisfies the neoclassical technology assumptions ($KA1$ and $KA2$)
- Dividing by (constant) workers L,

$$\tilde{y}(t) = A\tilde{k}(t)^{\alpha}$$

- FOC:

$$R(t) = \frac{\partial A\tilde{k}(t)^{\alpha}}{\partial \tilde{k}(t)} = \alpha A\tilde{k}(t)^{\alpha-1}$$

- From Euler's Theorem,

$$w(t) = \tilde{y}(t) - R(t)\tilde{k}(t) = (1-\alpha)A\tilde{k}(t)^{\alpha}$$

Equilibrium without Pop. Growth and Tech. Progress III

Example: Cobb-Douglas II

- Alternatively, same results w/ original Cobb-Douglas production function:

$$R(t) = \frac{\partial Y(t)}{\partial K(t)} = \alpha A K(t)^{\alpha-1} L^{1-\alpha}$$

- Defining the capital-labor ratio,

$$\tilde{k}(t) = \frac{K(t)}{L} \quad \Leftrightarrow \quad \boxed{K(t) = \tilde{k}(t) L}$$

- Replacing this expression in the first equation in this slide,

$$\begin{aligned} R(t) &= \alpha A K(t)^{\alpha-1} L^{1-\alpha} \\ &= \alpha A (\tilde{k}(t) L)^{\alpha-1} L^{1-\alpha} = \alpha A \tilde{k}(t)^{\alpha-1} \end{aligned}$$

- Also,

$$\begin{aligned} w(t) = \frac{\partial Y}{\partial L} &= (1-\alpha) A K(t)^{\alpha} L^{-\alpha} \\ &= (1-\alpha) A \tilde{k}(t)^{\alpha} \end{aligned}$$

which verifies Euler's Theorem

Equilibrium without Pop. Growth and Tech. Progress IV

- The per capita representation also allows re-writing the *fundamental law of motion* of Solow's growth model. To do so, divide both sides of equation (8) by L,

$$\tilde{k}(t+1) = (1-\delta)\tilde{k}(t) + sf(\tilde{k}(t)) \tag{11}$$

Definition of steady-state equilibrium

A steady-state equilibrium without technological progress and population growth is an equilibrium path in which $\tilde{k}(t) = \tilde{k}^*(t)$ for all t.

- The economy will approach this steady-state equilibrium over time
- In a steady-state equilibrium the capital-labor ratio remains constant!

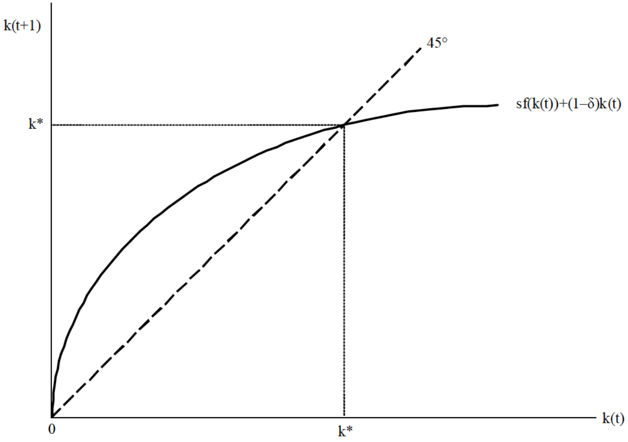


Figure: Visual representation of the steady state, mathematical characterization

Equilibrium without Pop. Growth and Tech. Progress VI

Steady-state Capital-labor Ratio II

- The curve represents equation (11) and the dashed line is the 45° line
- Their (positive) intersection gives the steady-state value of the capital-labor ratio $\tilde{k}^*$,

$$\frac{f(\tilde{k}^*)}{\tilde{k}^*} = \frac{\delta}{s} \tag{12}$$

- The other intersection is of no economic interest, and we ignore it!
- An alternative visual representation of the steady state is the intersection between $\delta\tilde{k}$ and the function $sf(\tilde{k})$
  - It depicts consumption and investment levels
  - It emphasizes that in this steady-state equilibrium: investment, $sf(\tilde{k})$, is set equal to the depreciated capital, $\delta\tilde{k}$

Equilibrium without Pop. Growth and Tech. Progress VII

Consumption and Investment in the Steady State

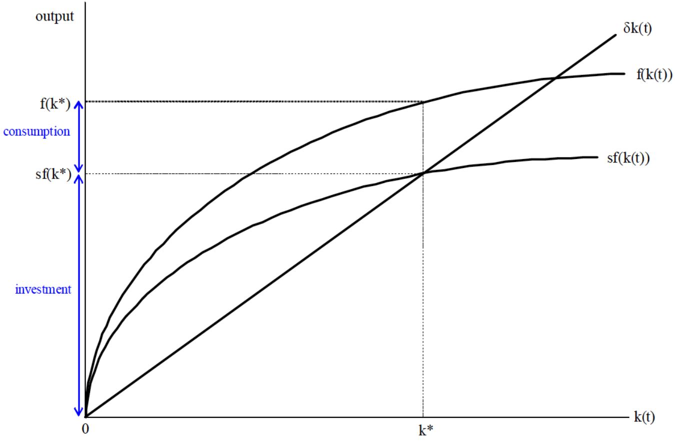


Figure: Visual representation of the steady state, economic characterization

Equilibrium without Pop. Growth and Tech. Progress VIII

Output and Consumption in the Steady State

Proposition 2

Consider the basic Solow growth model and suppose that the neoclassical technology assumptions $KA1$ and $KA2$ hold. Then there exists a unique steady-state equilibrium where the capital-labor ratio $\tilde{k}^* \in (0, \infty)$ is given by equation (11), per capita output is given by

$$\tilde{y}^* = f(\tilde{k}^*) \tag{13}$$

and per capita consumption is given by

$$\tilde{c}^* = (1 - s)f(\tilde{k}^*) \tag{14}$$

- Existence and uniqueness is guaranteed by Assumptions $KA1$ and $KA2$

- Panels A, B violate $KA2$ (Inada conditions), and panel C violates $KA1$

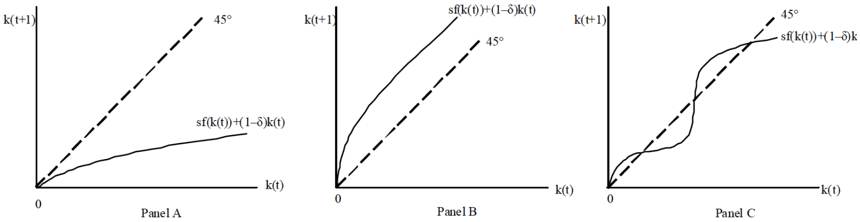


Figure: Equilibrium and production functions

Equilibrium without Pop. Growth and Tech. Progress X

- Countries with higher saving rates ($\uparrow s$) and better technologies ($\uparrow A$) will have higher capital-labor ratios and will be richer
- Countries with greater technological depreciation will tend to have lower capital-labor ratios and will be poorer
- The same applies to $\tilde{c}^*$ with respect to technologies A and saving rates s. Although there is a unique saving rate, s_{gold}, that maximizes steady-state consumption
- Write the steady state relationship between $\tilde{c}^*$ and s, suppressing other parameters:

$$\begin{aligned}\tilde{c}^* &= (1-s)f(\tilde{k}^*(s)) \\ &= f(\tilde{k}^*(s)) - \delta\tilde{k}^*(s)\end{aligned}$$

The second equality exploits that in the steady state $sf(\tilde{k}) = \delta\tilde{k}$

- Differentiating with respect to s,

$$\frac{\partial \tilde{c}^*(s)}{\partial s} = [f^{'}(\tilde{k}^*(s)) - \delta]\frac{\partial \tilde{k}^*}{\partial s} \tag{15}$$

- The golden rule saving rate s_{gold} must be consistent with $\partial \tilde{c}^*(s_{gold})/\partial s = 0$

Proposition 3

In the basic Solow growth model, the highest level of steady-state consumption is reached for s_{gold}, with the corresponding steady-state capital level k^*_{gold} such that

$$f^{'}(\tilde{k}^*_{gold}) = \delta \tag{16}$$

- When the economy is below $\tilde{k}^*_{gold}$, a higher saving rate will increase consumption; when the economy is above $\tilde{k}^*_{gold}$, steady-state consumption can be raised by saving less (*dynamic ineffiency*!)

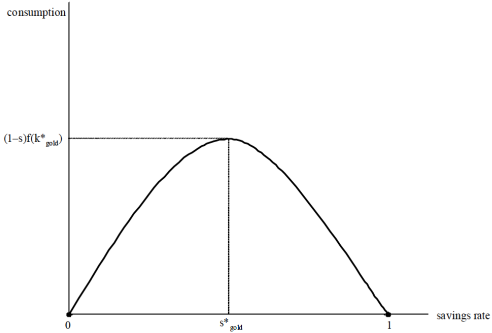
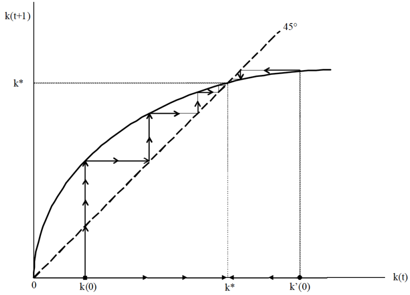


Figure: The Golden Rule

- Equilibrium path refers to the entire path of capital stock, output, consumption and factor prices (physics vs. economics)
- To see how the equilbrium path looks like we need to study the transitional dynamics of equation (11), starting with arbitrary capital-labor ratio, $\tilde{k}(0) > 0$
- We are often interested only in the steady state equilibrium!
- Let's look at transitional dynamics graphically...

Equilibrium without Pop. Growth and Tech. Progress XIV

Transitional Dynamics II



- **Capital deepening**: Starting at $\tilde{k}(0) < \tilde{k}^*$ the economy experiences growth until $\tilde{k}^*$, so that the capital-labor ratio increases (and also per capita income!)
- If the economy instead starts at $\tilde{k}' > \tilde{k}^*$, the economy would decumulate capital and experience negative growth!

- The basic Solow model has very nice properties:
  - Unique (stable) steady state
  - Simple comparative statics
- ... but so far has **no growth**: in the steady, there is no growth in the capital-labor ratio ($\tilde{k}^*$) and no growth in output per capita ($\tilde{y}^*$)!
- Solow model, without technological progress, can only explain growth during the transition phase (when $\tilde{k} < \tilde{k}^*$)
  - Eventually, growth slows down and comes to a halt!
- Although not in a desirable manner, the Solow model can account for sustained growth by introducing (*exogenous*) technological change!

The Solow Model in Continuous Time

- Instead of having time periods $t = 0, 1, \ldots$, we want to make time units as small as possible, i.e. we go to continuous time, $t \in \mathbb{R}_+$!
  - Most growth models formulated in continuous time
  - Analysis in continuous time, advantageous: flexibility, tractability, etc.

- We use the *dot notation* to denote time derivatives and thus to study the behavior of variable x from t to $t + 1$

$$\dot{x}(t) = \frac{dx(t)}{dt}$$

  $\dot{x}(t)$ is the continuous version of $x(t + 1) - x(t)$, i.e. the change in absolute terms in variable x

Solow Model with Pop. Growth and No Tech. Change I

- Nothing has changed in the production side, and the same equations apply! Now we simply think of variables as "instantaneous"
- In addition, let us now introduce population growth

$$L(t) = \exp(nt)L(0)$$

- Recall that,

$$\dot{K}(t) = sY(t) - \delta K(t)$$

and also that,

$$\tilde{k}(t) \equiv \frac{K(t)}{L(t)}$$

- By *taking logs and then derivatives* this last expression becomes,

$$\begin{aligned} \frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} &= \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)} \\ &= \frac{\dot{K}(t)}{K(t)} - n \end{aligned}$$

Solow Model with Pop. Growth and No Tech. Change II

where n is the population (*and also the labor force*) growth rate. For instance, when $n = 0.02$ the population grows at 2% per year

- Combining the last expression with the capital law of motion,

$$\begin{aligned}
\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} &= \frac{\dot{K}(t)}{K(t)} - n \\
&= \frac{sY(t) - \delta K(t)}{K(t)} - n \\
&= \frac{sY(t)}{K(t)} - \delta - n \\
&= \frac{sY(t)}{\tilde{k}(t)L(t)} - \delta - n \\
&= \frac{s\tilde{y}(t)}{\tilde{k}(t)} - \delta - n \qquad (17)
\end{aligned}$$

Solow Model with Pop. Growth and No Tech. Change III

- Multiplying both sides of equation (17) by $\tilde{k}(t)$ we can now express the change in capital per worker each period as,

$$\dot{\tilde{k}}(t) = s\tilde{y}(t) - (\delta + n)\tilde{k}(t) \tag{18}$$

- From equation (18) we observe that:
  - investment per worker ($s\tilde{y}$) increases capital per worker ($\tilde{k}$)
  - both depreciation (δ) and population growth (n) reduce $\tilde{k}$
- With these equations and making use of the definition of the steady state, we can now solve the model (i.e. to express endogenous variables as functions of parameters and exogenous variables):
  - Graphically
  - Analytically

Solow Model with Pop. Growth and No Tech. Change IV

Graphical Solution

- The line $(\delta + n)\tilde{k}$ represents the amount of investment per person required to keep the amount of capital per worker constant
- In the steady state the capital-labor ratio is constant, so that:

$$\dot{\tilde{k}} = 0 \Rightarrow s\tilde{y} = (\delta + n)\tilde{k}$$

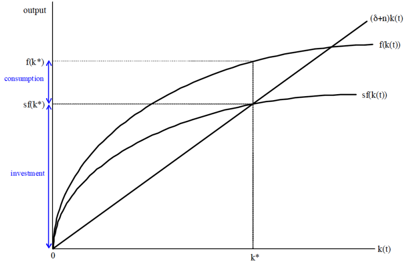


Figure: Graphical solution in Solow's model with population growth

- Recall that in the steady state the capital-labor ratio is constant, so that:

$$\dot{\tilde{k}} = 0 \Rightarrow s\tilde{y} = (\delta + n)\tilde{k}$$

- We can then solve the model for given production function in per worker terms, $\tilde{y}$

- In the simple case of the Cobb-Douglas production function, $\tilde{y}(t) = A\tilde{k}(t)^{\alpha}$, solving for $\tilde{k}^*$ we get:

$$\tilde{k}^* = \left(\frac{sA}{\delta + n}\right)^{\frac{1}{1-\alpha}}$$

- Having calculated $\tilde{k}^*$, we can now easily obtain the remaining variables of interest:
  - $\tilde{y}^*$ can now be expressed as a function of $\tilde{k}^*$,

$$\begin{aligned}\tilde{y}^* = f(\tilde{k}^*) &= A{\tilde{k}^*}^{\alpha} \\ &= A\left(\frac{sA}{\delta + n}\right)^{\frac{\alpha}{1-\alpha}} \\ &= A^{1/(1-\alpha)}\left(\frac{s}{\delta + n}\right)^{\frac{\alpha}{1-\alpha}}\end{aligned}$$

  - And $\tilde{c}^*$ as a function of $\tilde{y}^*$,

$$\begin{aligned}\tilde{c}^* &= (1-s)\tilde{y}^* \\ &= (1-s)A^{1/(1-\alpha)}\left(\frac{s}{\delta + n}\right)^{\frac{\alpha}{1-\alpha}}\end{aligned}$$

Solow Model with Pop. Growth and No Tech. Change VII

Take outs!

- Going from discrete to continuous time has not changed the basic features of the model!
- What does the Solow model tell us?
  - To replenish the capital-labor ratio, investment now must be determined by considering both depreciation and population growth (capital-labor ratio decreases as population grows!)
  - Richer countries are those with higher saving/investment rates, higher levels of technology, and lower depreciation- and population growth-rates!
- But the Solow model can still not explain sustained growth!
  - There is *no* per capita growth once the economy reaches the steady state: although output Y grows, output per worker $\tilde{y}$ does not!
  - So, eventually growth slows down and comes to a halt!

- Consider the capital accumulation equation (17):

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{s\tilde{y}(t)}{\tilde{k}(t)} - (\delta + n)$$

which in the Cobb-Douglas case can be rewritten as follows,

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = sA\tilde{k}(t)^{\alpha-1} - (\delta + n)$$

- For $\alpha < 1$: as $\tilde{k}$ rises, the growth rate of $\tilde{k}$ declines over time t
- Given that $\tilde{y}$ is proportional to $\tilde{k}$ ($\tilde{y}(t) = f(\tilde{k}(t))$), the same result applies to $\tilde{y}$

- The further the economy is below (above) the steady state $\tilde{k}^*$, the faster the economy grows (de-grows)

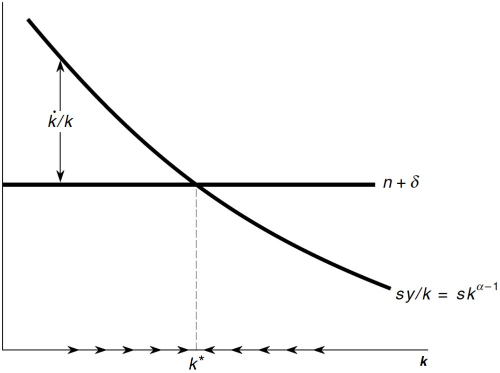


Figure: Transition dynamics in (Cobb-Douglas) Solow, no technological change (A normalized to 1)

- Sustained growth can be introduced into the Solow Model in several ways
- The simplest way to do so is to relax technology assumptions $KA1$ and $KA2$ so that $\alpha = 1$ in the Cobb-Douglas production function
- The resulting model is the so-called AK model,

$$Y(t) = F[K(t), L(t), A(t)] = AK(t) \tag{19}$$

where $A > 0$ is constant

- Assuming that population grows at constant rate n, the fundamental law of motion of the capital stock becomes,

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{K}(t)}{K(t)} - n = \frac{sY(t) - \delta K(t)}{K(t)} - n = sA - \delta - n$$

- There will be sustained growth in the capital-labor ratio and, therefore, in per capita output *iff* $sA - \delta - n > 0$
- The economy always grows at constant rate $sA - \delta - n$, regardless of the initial level of capital-labor ratio!

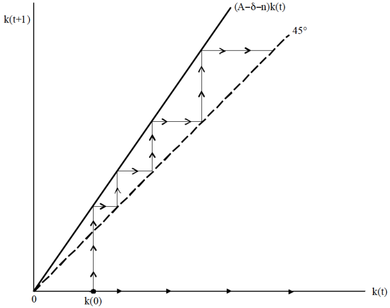


Figure: Sustained growth in the AK model

- The AK model, despite the advantage of its simplicity, presents many drawbacks:
  - Violates technology assumptions $KA1$ and $KA2$: it requires the production function to be linear in the capital stock
  - This implies that the share of income going to capital should equal 1, which contradicts empirical evidence
  - Most importantly, there is no technological progress, which is the major factor explaining the process of growth!
  - A satisfactory model of growth must thereof account for technological change!
- Next we introduce (*exogenous*) technological progress into the basic Solow model...
- ...although we will discuss AK models in more detail in Lecture 4!

Sustained Growth in the Solow Model I

Incorporating Technological Progress

- **Motivation**: with given input $\overline{X}$, society produces more output Y today than it did many years ago, $Y(\overline{X})_{today} > Y(\overline{X})_{1800}$
- In our model, changes in $A(t)$ capture improvements in technology
- In the Solow model we treat technological progress as exogenous ("*mana from heaven*"): economic agents cannot influence it!
- We thereof make the assumption that $A(t)$ grows at a constant rate g

$$A(t) = \exp(gt)A(0)$$

- This unrealistic assumption will be relaxed later on in this course
- But how to introduce technology $A(t)$ in the production function?
  - Consistent with *Kaldor facts*, i.e. **balanced growth path**: capital, output, consumption and population grow at constant rates so that the capital-output and capital-labor ratios are constant

Sustained Growth in the Solow Model II

Types of Neutral Technological Progress I

- **Hicks-neutral technological progress**

$$F[K(t), L(t), A(t)] = A(t)F[K(t), L(t)]$$

  - No change in the isoquants of the production function

- **Solow-neutral technological progress**

$$F[K(t), L(t), A(t)] = F[A(t)K(t), L(t)]$$

  - Capital-augmenting progress: isoquants slightly open downwards

- **Harrod-neutral technological progress**

$$F[K(t), L(t), A(t)] = F[K(t), A(t)L(t)]$$

  - Labor-augmenting progress: isoquants slightly open upwards

Sustained Growth in the Solow Model III

Types of Neutral Technological Progress II

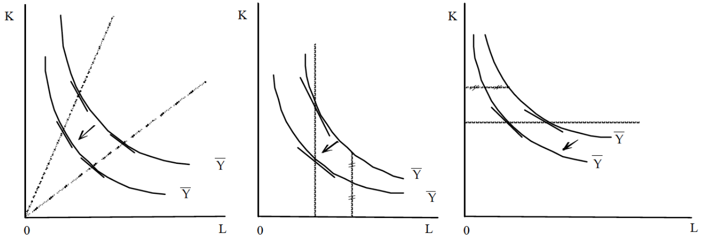


Figure: Hicks-neutral, Solow-neutral and Harrod-neutral shifts in isoquants.

- There are alternative ways to introduce neutral technological progress, e.g. vector-valued index of technology

$$F[K(t), L(t), \mathbf{A}(t)] = A_H(t) F[A_K(t) K(t), A_L(t) L(t)]$$

which are beyond the scope of this course

Sustained Growth in the Solow Model IV

- Although all types of technological progress are equally plausible *ex ante*, balanced growth in the long run is only possible under the labor-augmenting type

- Another important result that we take for granted (*Uzawa's Theorem*, 1961) establishes the constancy of factor shares along a *BGP*:

$$\alpha_K(t) \equiv \frac{R(t)K(t)}{Y(t)} \qquad \text{and} \qquad \alpha_L(t) \equiv \frac{w(t)L(t)}{Y(t)}$$

which, according to $KA1$ and Euler's theorem, $\alpha_K(t) + \alpha_L(t) = 1$

Sustained Growth in the Solow Model V

- Let's consider the Cobb-Douglas case with this specification

$$Y(t) = F[K(t), A(t)L(t)] = K(t)^{\alpha}(A(t)L(t))^{1-\alpha} \tag{20}$$

- Assume that population grows at rate n and further that there is technological progress at rate $g_A > 0$

$$\dot{A}(t) = A(t)g_A > 0$$

- Again, capital accumulates according to,

$$\dot{K}(t) = sF[K(t), A(t)L(t)] - \delta K(t) \tag{21}$$

- For simplicity we know express everything in *effective units of labor*,

$$\widehat{k}(t) \equiv \frac{K(t)}{A(t)L(t)}$$

- Taking logs and differentiating this expression wrt time,

$$\frac{\dot{\widehat{k}}(t)}{\widehat{k}(t)} = \frac{\dot{K}(t)}{K(t)} - g_A - n \tag{22}$$

Sustained Growth in the Solow Model VI

- Output per effective unit of labor can now be written as,

$$\begin{aligned}\widehat{y}(t) &\equiv \frac{Y(t)}{A(t)L(t)} \\ &= F\left[\frac{K(t)}{A(t)L(t)}, 1\right] \\ &\equiv f\left(\widehat{k}(t)\right)\end{aligned}$$

- Recall that income per capita is $\tilde{y}(t) \equiv Y(t)/L(t)$, so that

$$\begin{aligned}\tilde{y}(t) &= A(t)\widehat{y}(t) \\ &= A(t)f(\widehat{k}(t)) \qquad (23)\end{aligned}$$

- Now, even if $\widehat{y}(t)$ is constant, income per capita $\tilde{y}(t)$ grows over time because $A(t)$ is growing!

Sustained Growth in the Solow Model VII

- In models of technological change one should not look for steady states but rather for *balanced growth paths* where income per capita grows at a constant rate!

- Using the expression of Equation (21) and substituting into Eq. (22),

$$\frac{\dot{\widehat{k}}(t)}{\widehat{k}(t)} = \frac{sF[K(t), A(t)L(t)]}{K(t)} - (\delta + g_A + n)$$

- Using $\widehat{k}(t) \equiv K(t)/A(t)L(t)$,

$$\begin{aligned}\frac{\dot{\widehat{k}}(t)}{\widehat{k}(t)} &= \frac{sF[K(t), A(t)L(t)]}{\widehat{k}(t)A(t)L(t)} - (\delta + g_A + n) \\ &= \frac{sY(t)}{\widehat{k}(t)A(t)L(t)} - (\delta + g_A + n) \\ &= \frac{sf(\widehat{k}(t))}{\widehat{k}(t)} - (\delta + g_A + n) \qquad (24)\end{aligned}$$

Sustained Growth in the Solow Model VIII

- This result is very similar to the ones before, we have just incorporated g_A

- Equation (24) can now be rewritten as,

$$\dot{\widehat{k}}(t) = s\widehat{y}(t) - (\delta + g_A + n)\widehat{k}(t) \tag{25}$$

  which is the capital accumulation equation in terms of effective units of labor

- A steady state or *BGP* is now defined as an equilibrium in which the effective capital-labor ratio $\widehat{k}(t)$ is constant, i.e. $\dot{\widehat{k}}(t) = 0$

- We can now solve the model!

- If the economy is below (above) its steady state, the effective capital-labor ratio will rise (decline) gradually over time until $\widehat{k}^*$, where the economy grows along a balanced growth path!

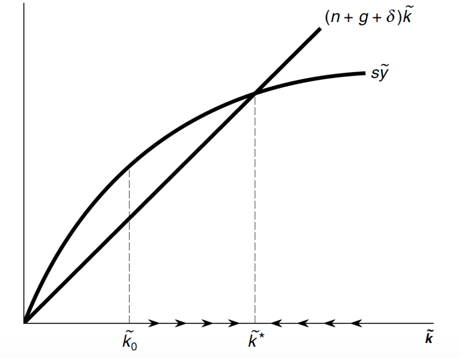


Figure: Solow model with technological progress

Sustained Growth in the Solow Model X

Analytical Solution I

- Remember that in the steady state, the effective capital-labor ratio is constant,

$$\dot{\widehat{k}}^*(t) = s\widehat{y}(t) - (\delta + g_A + n)\widehat{k}(t) = 0$$

which implies that

$$s\widehat{y}(t) = (\delta + g_A + n)\widehat{k}(t)$$

- In the particular case of the Cobb-Douglas production function,

$$s\widehat{k}(t)^\alpha = (\delta + g_A + n)\widehat{k}(t)$$

- Solving for $\widehat{k}^*$,

$$\widehat{k}^* = \left(\frac{s}{\delta + g_A + n}\right)^{1/(1-\alpha)}$$

- Substituting $\widehat{k}^*$ into the production function,

$$\widehat{y}^* = \left(\frac{s}{\delta + g_A + n} \right)^{\alpha/(1-\alpha)}$$

- To better appreciate the role of technology recall equation (23) and think of $\tilde{y}^*(t)$ as a function of $\widehat{y}^*$,

$$\begin{aligned} \tilde{y}^*(t) &= A(t)\widehat{y}^* \\ &= A(t) \left(\frac{s}{\delta + g_A + n} \right)^{\alpha/(1-\alpha)} \end{aligned} \tag{26}$$

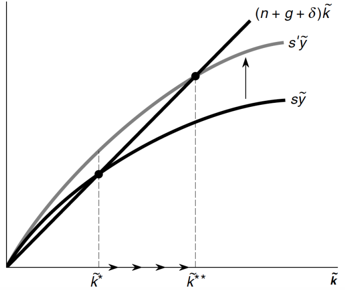
- Clearly, $\tilde{y}^*(t)$ depends on technology A and time t

- Changes in the investment- or population growth- rate affect the long-run *level* of output per worker but *not* the long-rung *growth rate* of output per worker!

Sustained Growth in the Solow Model XI

Comparative Statistics I

- **Shocks to the investment rate**: an increase (decrease) from s to s' moves the economy to a higher (lower) steady state $\widehat{k}^{**}$
  - At the initial $\widehat{k}^{*}$, investment exceeds the amount needed to keep $\widehat{k}^{*}$ constant so that $\widehat{k}$ begins to rise!
  - The increase in s raises the growth rate *temporarily* (along the transition to $\widehat{k}^{**}$)



Sustained Growth in the Solow Model XII

Comparative Statistics II

- Prior to the increase in s, output per worker grew at rate g; at the time of the increase t^*, $\tilde{y}$ grows more rapidly until the economy reaches the new steady state, when the growth rate returns to g!

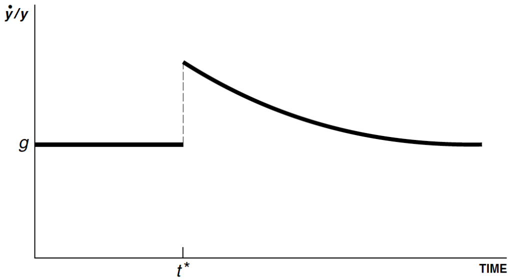


Figure: Effect of an increase in s on growth

Sustained Growth in the Solow Model XIII

Comparative Statistics III

- Although policy changes in the Solow model have no long-run *growth effect*, they can have *level effects*!

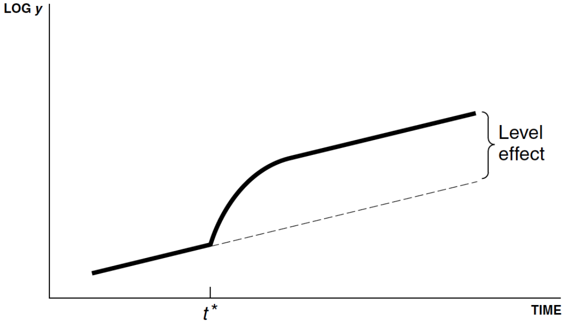


Figure: Level effects in the Solow framework

Sustained Growth in the Solow Model XIV

Comparative Statistics IV

- **Shocks to the population growth rate**: if the population growth rate rises (declines) from n to $n^{'}$, the economy reaches a new lower (higher) steady state $\widehat{k}^{**}$
  - At the initial $\widehat{k}^{*}$, investment per effective worker is no longer high enough to keep $\widehat{k}^{*}$ constant!

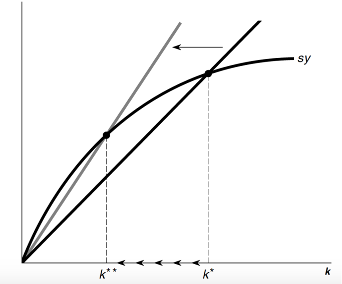


Figure: An increase in the population growth rate

Evaluation of the Solow Model

- Solow's model allows us to study capital accumulation and the implications of technological change
  - Without technological progress there is no sustained growth
- We have, however, learned little about technological progress
  - It is exogenous, "manna from heaven", a black box, not influenced by economics agents!
  - We need a model with endogenous technological progress: what makes some countries to develop better technologies?
- Even on the question of capital accumulation the Solow model is not entirely satisfactory
  - Capital accumulation is determined by the savings-, depreciation- and population growth- rate (*all exogenous*!)
- Overall, the key take outs from the Solow model are that: 1) to understand growth we must understand the processes of capital accumulation and technological progress; and 2) government intervention may have *level*- but no *long-run growth*- effects

Convergence I

- We did previously establish that the further below an economy is from its steady state, the faster it grows (and viceversa)
- Now we look in more detail at this result and try to generalize it for a cross section of countries
- Recall that,

$$g_{\tilde{k}}(t) \equiv \frac{s\tilde{y}(t)}{\tilde{k}(t)} - (\delta + n)$$
$$= sf(\tilde{k}(t))\tilde{k}(t)^{-1} - (\delta + n)$$

- The derivative of $g_{\tilde{k}}$ *wrt* $\tilde{k}$,

$$\frac{\partial g_{\tilde{k}}}{\tilde{k}} = sf^{'}(\tilde{k})\tilde{k}^{-1} - sf(\tilde{k})\tilde{k}^{-2}$$
$$= \frac{s}{\tilde{k}}\left(f^{'}(\tilde{k}) - \frac{f(\tilde{k})}{\tilde{k}}\right) < 0$$

Convergence II

- This result,

$$\frac{\partial g_{\tilde{k}}}{\partial \tilde{k}} = \frac{s}{\tilde{k}} \left(f^{'}(\tilde{k}) - \frac{f(\tilde{k})}{\tilde{k}} \right) < 0$$

  implies that smaller values of $\tilde{k}$ are associated with larger values of $g_{\tilde{k}}$

- Does this result mean that:
  - economies with lower capital per worker tend to grow faster?;
  - or, more subtly, that there is convergence across economies?
- The answer to these questions is that **it depends**!
  - If economies are structurally similar (same values for s, n, δ, access to the same technology A and production function $f(\cdot)$), then yes: they share the same steady-state values for $\tilde{k}^*$ and $\tilde{y}^*$
  - If economies are *not* structurally similar, then economies do not need to converge as they have different steady-state values
- Let's look at these two cases more closely!

Convergence III

- Assume two economies, *rich* and *poor*, are structurally similar and that the only difference between them is in $\tilde{k}(0)$ such that $\tilde{k}_{rich}(0) > \tilde{k}_{poor}(0) > 0$. Then, the model predicts that the less-advanced economy will exhibit higher growth rates $g_{\tilde{k}}$

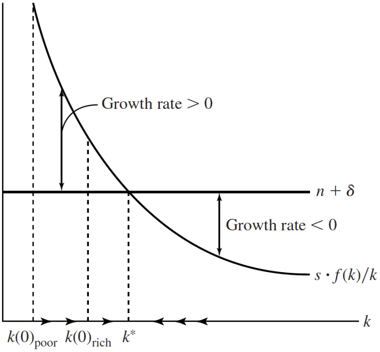


Figure: Transitional dynamics in the Solow model without technological progress

Convergence IV

- This result suggests a form of convergence: **conditional convergence**
  - Provided that a group of countries or regions are structurally similar, the economies with lower starting values of the capital/labor ratio will have higher per capita growth rates and tend thereby to catch up with, or converge to, the initially-richer economies
- Another, less-restrictive form of convergence: **absolute convergence**
  - Poorer countries or regions tend to grow faster than richer ones regardless even without conditioning on the steady-state determinants
- Because absolute convergence is less restrictive than conditional convergence, it is harder to observe in practice

Convergence V

- If economies are *not* structurally similar they may have different steady states
  - if the rich economy is further away from its own steady state than the poor one, then we expect $g_{\tilde{y}}^{rich} > g_{\tilde{y}}^{poor} > 0$ and divergence rather than (*absolute*) convergence

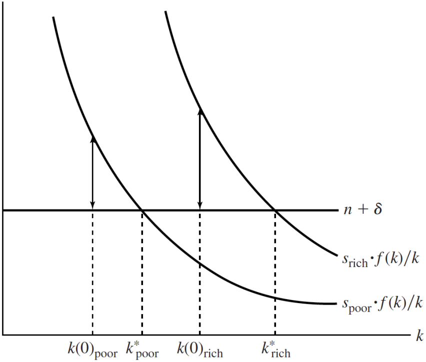


Figure: One example of why absolute convergence may not hold

- Does the data support the existence of conditional- and absolute-convergence?
  - No evidence of absolute convergence but rather of conditional one

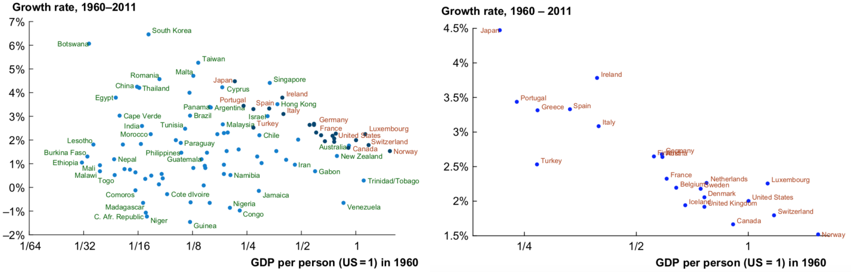


Figure: (a) World economies (b) OECD economies

Convergence VII

Further Evidence of Conditional Convergence: US States

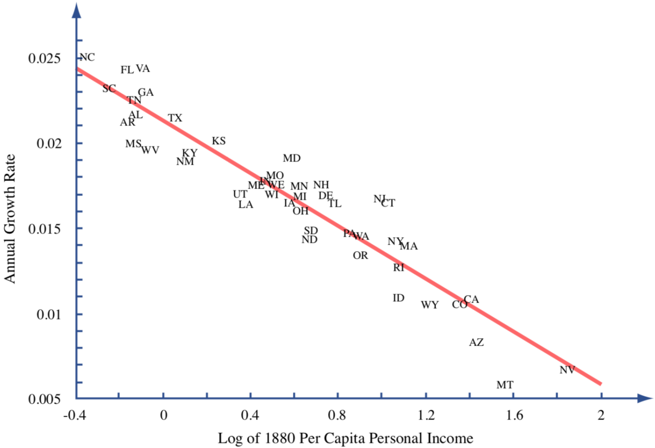


Figure: Convergence across US States, 1880-2000

Convergence VIII

Further Evidence of Conditional Convergence: Japanese Prefectures

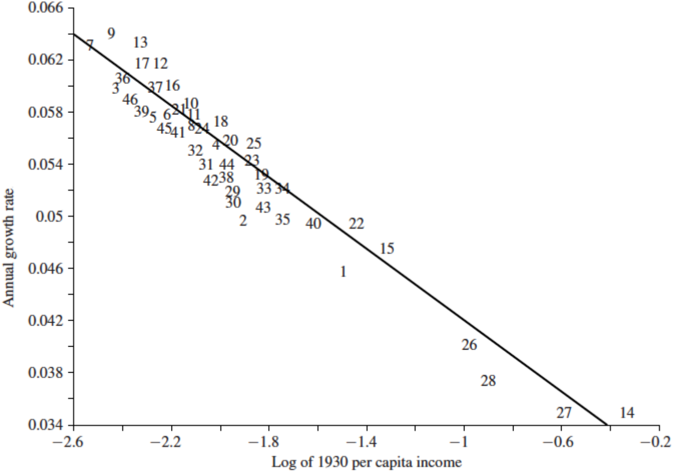


Figure: Convergence across Japanese Prefectures, 1930-90

Convergence IX

- Next, we establish formally the tendency of an economy towards its own steady state

- For this, recall that in the steady state the capital-labor ratio is constant so that,

$$\dot{\tilde{k}} = sf(\tilde{k}^*) - (\delta + n)\tilde{k}^* = 0 \quad \Rightarrow \quad sf(\tilde{k}^*) = (\delta + n)\tilde{k}^*$$

- Solving for s,

$$s = (\delta + n)\frac{\tilde{k}^*}{f(\tilde{k}^*)}$$

- Plugging in this expression in $g_{\tilde{k}}$,

$$\begin{aligned} g_{\tilde{k}} &= \frac{sf(\tilde{k})}{\tilde{k}} - (\delta + n) \\ &= (\delta + n)\left[\frac{f(\tilde{k})/\tilde{k}}{f(\tilde{k}^*)/\tilde{k}^*} - 1\right] \end{aligned}$$

Convergence X

- From this expression,

$$g_{\tilde{k}} = (\delta + n)\left[\frac{f(\tilde{k})/\tilde{k}}{f(\tilde{k}^*)/\tilde{k}^*} - 1\right]$$

is clear that the growth rate of the capital-labor ratio depends on where the economy is currently at *wrt* its steady-state equilibrium

1. if $\tilde{k} < \tilde{k}^*$, then $g_{\tilde{k}} > 0$
2. if $\tilde{k} = \tilde{k}^*$, then $g_{\tilde{k}} = 0$
3. if $\tilde{k} > \tilde{k}^*$, then $g_{\tilde{k}} < 0$

The Solow Model and the Data I

Growth Accounting I

- Multiple ways to look at the data through the Solow Model:
  1. Growth accounting exercises: Solow's (1957) contribution
  2. Regression-based approaches
  3. Calibration exercises
  4. Other advanced approaches (*beyond the scope of this course*!)

- Solow's (1957) growth accounting exercise starts with

$$Y(t) = A(t)K(t)^{\alpha}L(t)^{1-\alpha}$$

- Taking logs and differentiating,

$$\begin{aligned}\frac{\dot{Y}(t)}{Y(t)} &= \frac{\dot{A}(t)}{A(t)} + \alpha\frac{\dot{K}(t)}{K(t)} + (1-\alpha)\frac{\dot{L}(t)}{L(t)} \\ &= g_A(t) + \alpha g_K(t) + (1-\alpha)n \qquad (27)\end{aligned}$$

- Output growth is equal to the growth rate of technological progress plus a weighted average of the growth rates of capital and labor

- Eq. (27) is the fundamental growth accounting equation and it can be applied to understand the contribution of the different factors to the process of growth (*as in ME2720*!)
- Factor shares usually calculated as averages of beginning-of-period and end-of-period values
- Technology's contribution, $g_A(t)$, calculated as the residual and referred to as TFP or multi-factor productivity
- Given our interest in growth in output per worker we can subtract $\dot{L}(t)/L(t)$ from both sides of equation (27),

$$\frac{\dot{\tilde{y}}(t)}{\tilde{y}(t)} = \alpha \frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} + \frac{\dot{A}(t)}{A(t)} = \alpha g_{\tilde{k}}(t) + g_A(t) \quad (28)$$

- Output per capita growth is equal to the contribution of physical capital per worker plus the contribution of technological progress

The Solow Model and the Data III

Growth Accounting III

- Solow's accounting exercise revealed that technology could explain a large part of the growth process (we verified this in *ME2720*!)

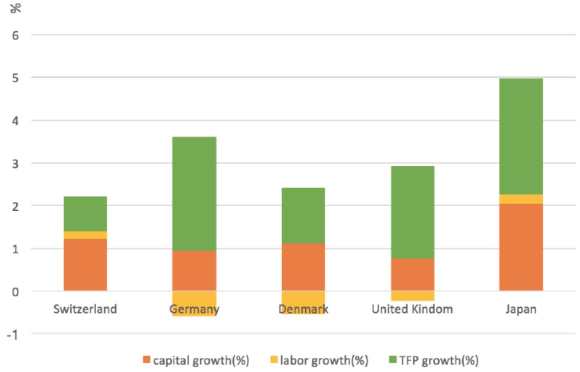


Figure: Growth accounting, 1970-1990 (Wang et al., 2017)

The Solow Model and the Data IV

Growth Accounting IV

- Although there has lately been substantial and sustained *TFP* growth, there are also slow downs (as in the 1970s!)
  - Introduction of GPTs (e.g. electricity, internet)
  - Changing composition of the labor force (e.g. from agriculture to manufacturing)
  - Less resources devoted to R&D
  - ...

- Growth accounting exercises to calculate the importance of technology have been however widely criticized:
  - The residual as "measure of our ignorance" (Abramovitz, 1957)
  - Underestimation of inputs (labor hours vs. effective labor hours & physical capital prices not capturing technology improvements) would overestimate the importance of technology
  - Underestimation of $g_Y(t)$ would biased downwards the contribution of $g_A(t)$

The Solow Model and the Data V

Regression Analyses I

- Regression analysis pioneered by Barro (1991) and Barro and Sala-i-Martin (1992, 2004)
- Mathematical arrangements of Solow's fundamental equations can yield estimating equations that also allow to test for convergence:

$$\frac{\dot{\tilde{y}}(t)}{\tilde{y}(t)} = g_A - (1-\alpha)(\delta + g_A + n)(\log \tilde{y}(t) - \log \tilde{y}^*(t))$$

  where the term $(1-\alpha)(\delta + g_A + n)$ multiplying $\log \tilde{y}(t) - \log \tilde{y}^*(t)$ captures the speed of convergence

- Plausible values for parameteres ($g_A = 0.02; n = 0.01; \delta = 0.05;$ $\alpha = 1/3$) suggests that the convergence coefficient should be about 0.054 ($\approx 0.67 \times 0.08$); a speed of convergence much faster than what's witnessed in the data!

The Solow Model and the Data VI

Regression Analyses II

- Simple regression equations are usually of the form:

$$g_{i,t,t-1} = \alpha + \beta \log \tilde{y}_{i,t-1} + \varepsilon_{i,t}$$

- When this equation is estimated in OECD economics, β is negative, providing evidence for convergence...
- ... but when estimation is performed in the sample of all economies, there is no evidence for convergence (if any, for divergence: $\beta > 0$)
- The notion of absolute convergence is in general too demanding: it requires income gaps to narrow over time despite the fact that countries differ in terms of technological opportunities, investment behavior, institutions, and so on!

The Solow Model and the Data VII

- Literature favors the concept of conditional convergence: β should be negative once we control for other relevant characteristics (e.g. schooling, fertility, investment, government role, etc.):

$$g_{i,t,t-1} = \mathbf{X}_{i,t}^{T}\gamma + \beta \log \tilde{y}_{i,t-1} + \varepsilon_{i,t}$$

where $\mathbf{X}_{i,t}^{T}$ includes the relevant characteristics mentioned above, as well as a constant

- Recently growth regressions have moved from cross-country- to panel data- approaches like:

$$\log \tilde{y}_{i,t} = \mathbf{X}_{i,t}^{T}\gamma + \beta \log \tilde{y}_{i,t-1} + \mu_i + \xi_t + \varepsilon_{i,t}$$

  - PD reduces the risk of ommitted-variable bias: inclusion of μ_i
  - PD reduces the risk of simultaneity bias: inclusion of ξ_t
  - ...

The Solow Model and the Data VIII

Regression Analyses IV

- The simple implementation of regressions, together with its ability to easily bridge theory and data, made them popular
- Regression analysis however also suffer from pitfalls:
  - Endogeneity: almost all variables in γ are endogeneous, rendering estimates of β to be inconsistent
  - Reverse causality
  - Ommitted-variable biased and simultaneity bias may remain despite the inclusion of fixed effects
  - ...

The Solow Model and the Data IX

Calibration Exercises I

- Instead of estimating productivity differences, we can calibrate the term *TFP* (Klenow and Rodriguez, 1997; Hall and Jones, 1999)
- Advantage of calibration is that ommitted-variable bias is less of a concern but assumptions of functional forms are much more important
- Consider the functional form:

$$Y_j = K_j^{\alpha}(A_j H_j)^{1-\alpha}$$

  where H_j is the stock of human capital in country j, capturing the amount of efficiency units of labor available (*more later*!)

- Given series for H_j, K_j and α, we can "predict" incomes at a point in time,

$$\widehat{Y}_j = K^{1/3}(A_x H_j)^{2/3}$$

  where A_x is the level of labor-augmenting technology in country x, which will be taken as benchmark

- Once $\widehat{Y}_j$'s have been constructed, we can compare them to actual series
- Gaps between series can be thought of as technology's contribution!
- These exercises usually reveal that:
  - differences in physical capital and human capital still matter a great deal!
  - technology differences are *extremely* important!
- A major challenge in calibration exercises is that factor shares are not available and α cannot be calculated; usually imposing $\alpha = 1/3$
- Also, calibration exercises are quite unflexible in terms of functional forms, assumptions regarding physical and human capital across countries, imposition of no human capital externalities, etc.

Augmented Solow Model I

- There are multiple ways to augment the Solow Model: human capital, government role, international trade, etc.

- Mankiw, Romer and Weil (1992) did however emphasize the role of human capital which, allegedly, could improve the fit of the model

- Recall that with human capital we aim to capture labor-related, producitivity-enhancing characteristics (e.g. skills, education, know-how, etc.)
  - Labor supplied by different individuals is not equally productive neither *within* professions (e.g. RA vs. Professor) nor *across* professions (e.g. bartender vs. engineer)

- Human capital can be introduced into the (aggregate) production function in two ways:
  1. Additional term: $Y = F[K, H, AL]$
  2. As a single term factoring in labor: $Y = F[K, AH]$, and $H = \exp(\psi u)L$

Augmented Solow Model II

- More micro-founded models introduce human capital as a term factoring in labor

- For our purposes, as an initial step to gain understanding, we will consider labor as an additional term so that

$$Y(t) = F[K(t), H(t), A(t)L(t)] \tag{29}$$

- $KA1$ and $KA2$ are modified in such a way that "*neoclassical technology assumptions*" still hold for the new production function

Augmented Solow Model III

KA1$'$: Continuity, Differentiability, Positive & Diminishing MPs, CRS

The production function $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ in Eq. (29) is twice continuously differentiable in K, H and L, and satisfies

$$F_K(K, H, AL) > 0, \qquad F_L(K, H, AL) > 0, \qquad F_L(K, H, AL) > 0$$

$$F_{KK}(K, H, AL) < 0, \qquad F_{HH}(K, H, AL) < 0, F_{LL}(K, H, AL) < 0$$

Moreover, F exhibits constant returns to scale in its three arguments.

KA2$'$: F satisfies Inada (1961) Conditions

$$\lim_{K \to 0} F_K(K, H, AL) = \infty \quad \text{and} \quad \lim_{K \to \infty} F_K(K, H, AL) = 0, \quad \forall H, AL > 0$$

$$\lim_{H \to 0} F_L(K, H, AL) = \infty \quad \text{and} \quad \lim_{H \to \infty} F_L(K, H, AL) = 0, \quad \forall K, AL > 0$$

$$\lim_{L \to 0} F_L(K, H, AL) = \infty \quad \text{and} \quad \lim_{L \to \infty} F_L(K, H, AL) = 0, \quad \forall K, H, A > 0$$

Augmented Solow Model IV

- Investments in human capital take a similar form to investments in physical capital:
  - households save a fraction $s_k \in (0, 1)$ of their disposable income to invest in K;
  - and a fraction $s_h \in (0, 1)$ to invest in H
- Human capital also depreciates in a similar manner as physical capital, and we denote δ_h and δ_k, respectively
- We keep assumptions regarding population growth and labor-augmenting technological progress:
  - $\dot{L}(t)/L(t) = n$
  - $\dot{A}(t)/A(t) = g_A$
- We now define effective human and capital ratios as,

$$\widehat{k}(t) \equiv \frac{K(t)}{A(t)L(t)} \qquad \widehat{h}(t) \equiv \frac{H(t)}{A(t)L(t)}$$

Augmented Solow Model V

- Output per effective unit of labor can be written as,

$$\begin{aligned}\hat{y}(t) &\equiv \frac{Y(t)}{A(t)L(t)} \\ &= F\left(\frac{K(t)}{A(t)L(t)}, \frac{H(t)}{A(t)L(t)}, 1\right) \\ &\equiv f(\hat{k}(t), \hat{h}(t)) \end{aligned} \tag{30}$$

- There are now laws of motion for both physical and human capital:

$$\dot{\hat{k}}(t) = s_k f(\hat{k}(t), \hat{h}(t)) - (\delta_k + g_A + n)\hat{k}(t) \tag{31}$$

$$\dot{\hat{h}}(t) = s_h f(\hat{k}(t), \hat{h}(t)) - (\delta_h + g_A + n)\hat{h}(t) \tag{32}$$

Augmented Solow Model VI

- A steady-state equilibrium is now defined by effective human- and physical- capital ratios, $(\widehat{k}^*, \widehat{h}^*)$

- Remember that in the steady-state the effective human- and physical ratios must be constant so that,

$$\dot{\widehat{k}}(t) = s_k f(\widehat{k}(t), \widehat{h}(t)) - (\delta_k + g_A + n)\widehat{k}(t) = 0$$
$$\dot{\widehat{h}}(t) = s_h f(\widehat{k}(t), \widehat{h}(t)) - (\delta_h + g_A + n)\widehat{h}(t) = 0 \quad (33)$$

- As in the basic Solow model we discard the (trivial) steady state associated with $f(0,0)$ and require that $\widehat{k}^*, \widehat{h}^* > 0$

- Also, as in the basic Solow model, there exists a unique steady state

Augmented Solow Model VII

Cobb-Douglas Setting I

- We now consider the augmented Solow model in the Cobb-Douglas production setting
- Consider the (aggregate) production function of the form,

$$Y(t) = K(t)^{\alpha} H(t)^{\beta} (A(t)L(t))^{1-\alpha-\beta} \tag{34}$$

  where $0 < \alpha, \beta < 1$ and $\alpha + \beta < 1$
- Note that this production function nests to the basic Solow model if $\beta = 0$
- Recalling the definition of $\widehat{y}(t), \widehat{k}(t)$ and $\widehat{h}(t)$, production function $Y(t)$ can easily be expressed in terms of effective units of labor as,

$$\widehat{y}(t) = \widehat{k}(t)^{\alpha} \widehat{h}(t)^{\beta} \tag{35}$$

- Making use of the steady-state conditions (equations in 33), we can obtain equilibrium values for $\widehat{k}^*$ and $\widehat{h}^*$:

$$\widehat{k}^* = \left(\left(\frac{s_k}{\delta_k + g_A + n} \right)^{1-\beta} \left(\frac{s_h}{\delta_h + g_A + n} \right)^{\beta} \right)^{\frac{1}{1-\alpha-\beta}}$$

$$\widehat{h}^* = \left(\left(\frac{s_k}{\delta_k + g_A + n} \right)^{\alpha} \left(\frac{s_h}{\delta_h + g_A + n} \right)^{1-\alpha} \right)^{\frac{1}{1-\alpha-\beta}} \tag{36}$$

  To verify these equilibrium values is left as an exercise for Assign. 1!

- Equations in (36) tell us that higher saving rates s_k do not only increase $\widehat{k}^*$ but also $\widehat{h}^*$
  - → The same applies for s_h!

- Also, given equations in (36), we can express output per effective unit of labor as,

$$\widehat{y}^* = \widehat{k}^{*\alpha} \widehat{h}^{*\beta}$$
$$= \left(\frac{s_k}{\delta_k + g_A + n} \right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{s_h}{\delta_h + g_A + n} \right)^{\frac{\beta}{1-\alpha-\beta}} \quad (37)$$

- The relative contributions of the savings rates in physical and human capital on output per effective unit of labor depend on the shares of physical and human capital, α and β, respectively
  - $\alpha > \beta \Rightarrow s_k$ is more important than s_h
  - $\alpha < \beta \Rightarrow s_h$ is more important than s_k
  - $\alpha = \beta \Rightarrow s_h$ and s_k are equally important

- If we want to express output in per capita terms and emphasize the importance of technology, simply multiply Eq. (37) by $A(t)$

$$\begin{aligned}\tilde{y}(t)^* &= A(t)\widehat{y}(t)^* \\ &= A(t)\left(\frac{s_k}{\delta_k + g_A + n}\right)^{\frac{\alpha}{1-\alpha-\beta}}\left(\frac{s_h}{\delta_h + g_A + n}\right)^{\frac{\beta}{1-\alpha-\beta}} \qquad (38)\end{aligned}$$

- This framework can be generalized to compare the state of affairs in different countries,

$$\tilde{y}_j^*(t) \equiv \frac{Y(t)}{L(t)} = A_j(t)\left(\frac{s_{k,j}}{\delta_k + g_{A,j} + n_j}\right)^{\frac{\alpha}{1-\alpha-\beta}}\left(\frac{s_{h,j}}{\delta_h + g_{A,j} + n_j}\right)^{\frac{\beta}{1-\alpha-\beta}} \qquad (39)$$

- Assuming $g_{A,j}$ differ across countries, income per capita will diverge given that $A_j(t)$ will growth at different rates

- This approach is what Mankiw, Romer and Weil (1992) used for regression analysis in their seminal paper

- MRW assume that technological know-how in all countries grows at the common rate g_A but that initial levels of technology differ across countries

$$A_j(t) = \overline{A}_j \exp(gt)$$

- Taking logs in Eq. (39) they obtain the following log-linear equation:

$$\log \widehat{y}_j^*(t) = \log \overline{A}_j + g_A t + \frac{\alpha}{1-\alpha-\beta}\left(\frac{s_{k,j}}{\delta_k + g_A + n_j}\right) + \frac{\beta}{1-\alpha-\beta}\log\left(\frac{s_{h,j}}{\delta_h + g_A + n_j}\right) \tag{40}$$

which can be easily estimated with cross-country data

Augmented Solow Model XII

Cobb-Douglas Setting VI: MRW (1992)

- By applying eq. (40) we can uncover the values of α and β
- MRW assume that:
  - $\delta_k = \delta_h = \delta$
  - $\delta + g_A = 0.05$
  - $s_{k,j}$ is approximated with average investment rates
  - $s_{h,j}$ is taken as the fraction of the school-aged population enrolled in secondary education
  - n_j computation is straightforward
- If $\overline{A}_j$ is correlated with investment rates $s_{k,j}$ and $s_{h,j}$, Eq. (40) would yield inconsistent estimates. . .
  - . . . so MRW crucially assume "*orthogonal technology*", i.e. $\overline{A}_j = \varepsilon_j A$ with ε_j orthogonal to all other variables, which allows consistent estimation

Augmented Solow Model XIII

Cobb-Douglas Setting VII: MRW's (1992) Basic Solow Model Estimates

- MRW's (1992) estimation of the basic Solow model, i.e. $\beta = 0$

Table: Estimates of the Solow Model. **Notes**. ***, significance at the 1%.

	MRW (1985)	**Acemoglu** (1985), updated	**Acemoglu** (2000)
$\log(s_k)$	1.42^{***}	1.01^{***}	1.22^{***}
$\log(n + g_A + \delta)$	-1.97^{***}	-1.12^{***}	-1.31^{***}
Adjusted R^2	.59	.49	.49
Implied α	.59	.50	.55
Obs.	98	98	107

- The coefficient 1.42 for $\alpha/(1-\alpha)$ suggests that $\alpha \approx 2/3$ but data tell us $\alpha \approx 1/3$

Augmented Solow Model XIII

Cobb-Douglas Setting VIII: MRW's (1992) Augmented Solow Model Estimates

Table: Estimates of the Augmented Solow Model. **Notes**. ***, significance at the 1%.

	MRW (1985)	**Acemoglu** (1985), updated	**Acemoglu** (2000)
$\log(s_k)$	0.69^{***}	0.65^{***}	0.96^{***}
$\log(n + g_A + \delta)$	$-.73^{***}$	-1.02^{***}	-1.06^{***}
$\log(s_h)$	0.66^{***}	0.47^{***}	0.70^{***}
Adjusted R^2	.78	.65	.60
Implied α	.30	.31	.36
Implied β	.28	.22	.26
Obs.	98	98	107

- Adjusted R^2 values significantly increase w.r.t. basic Solow estimates, and α values are now approx. $1/3$!
- Augmented Solow much better fits the data, and suggests that about 70% of cross-country income differences are due to physical and human capital!

Summarizing I

What have we learned?

- **Solow's model** is one of the first workhorse models in the growth literature; it **helps us to understand the mechanics of growth**
- At the center of the Solow model there is the neoclassical production function, allowing us to bridge theory with empirics
- The Solow model can run in both discrete- and continuous- time
- The Solow model is a **blend of Keynesian and neoclassical models**
- The Solow model can be augmented to incorporate relevant factors such as human capital
- Despite its simplicity, the Solow model sheds light on the importance of saving/investment rates, population growth, human capital and technology differences

Summarizing II

What have we learned?

- The Solow model allows comparative statistics and gives an intuition of why some countries could be poor whilst others rich
- The Solow model can account for both **convergence** and **divergence**
  - conditional- vs. absolute- convergence
- The Solow model has room for public policy!
- Overall, the Solow model is **not entirely satisfactory**:
  - the most important variables are exogenous ("*manna from heaven*")
  - Solow's model emphasizes the proximate causes of growth...
  - ...but to say that a country is poor because it has little physical- and human- capital and inefficient technology is like to say that a person is poor because it has no money!
  - There are factors that make a country to have more physical- and human- capital and more efficient technologies (as there are factors that make a person to have more money than others)

Summarizing III

What have we learned?

- A satisfactory theory of economic growth that allows us to understand cross-country income differences requires both an analysis of the *proximate* as well as *fundamental* causes of economic growth
- To analyze the *proximate* causes of growth is essential to understand the mechanics of growth, and crucial to bear in mind when designing useful models
- By studying the *fundamental* causes of growth we can understand why some societies make choices that lead them to low physical- and human- capital bundles as well as to have inefficient technologies, eventually ending up in relative poverty
- Next, we move to endogenous growth theory, which aims to build further on Solow and also to get rid of some of its shortcomings!!

Thank you for your attention!